

# Isomorphism spectra and computably composite structures

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**Abstract.** Adapting a result of Bazhenov, Kalimullin & Yamaleev [4], we show that if a Turing degree  $\mathbf{d}$  is the degree of categoricity of a computable structure  $\mathcal{M}$  and is not the strong degree of categoricity of any computable structure, then  $\mathcal{M}$  has a pair of computable copies whose isomorphism spectrum is not finitely generated. Motivated by this result, we introduce a class of computable structures, called *computably composite structures*, with the property that the isomorphisms between arbitrary computable copies of these structures are exactly the unions of isomorphisms between the computable copies of their components. We use this to show that any computable union of isomorphism spectra is also an isomorphism spectrum. In particular, this gives examples of isomorphism spectra that are not finitely generated.

## 1 Introduction

Mathematical structures are usually considered up to isomorphism. However, isomorphic structures may have different computability-theoretic properties. In particular, a pair of isomorphic computable structures need not have any computable isomorphisms. A major topic in computable structure theory is the complexity of isomorphisms between isomorphic computable structures. See [11] for background in computability theory, and [1; 10] for background in computable structure theory.

If  $\mathcal{A}$  and  $\mathcal{B}$  are computable structures, we write  $f: \mathcal{A} \cong \mathcal{B}$  to mean that  $f$  is an isomorphism from  $\mathcal{A}$  to  $\mathcal{B}$ . The *isomorphism spectrum* of  $\mathcal{A}$  and  $\mathcal{B}$ , denoted by  $\text{IsoSpec}(\mathcal{A}, \mathcal{B})$ , is the upward-closed set of Turing degrees that compute some isomorphism  $f: \mathcal{A} \cong \mathcal{B}$ ; that is,  $\text{IsoSpec}(\mathcal{A}, \mathcal{B}) = \{\mathbf{d} : \exists f: \mathcal{A} \cong \mathcal{B}[f \leq_T \mathbf{d}]\}$ . A structure  $\mathcal{A}$  is  *$\mathbf{d}$ -computably categorical* if  $\mathbf{d} \in \text{IsoSpec}(\mathcal{A}, \mathcal{B})$  for every computable copy  $\mathcal{B}$  of  $\mathcal{A}$ . We say that  $\mathcal{A}$  is *computably categorical* if it is  $\mathbf{0}$ -computably categorical. The *categoricity spectrum* of  $\mathcal{A}$ , denoted by  $\text{CatSpec}(\mathcal{A})$ , is the upward-closed set of Turing degrees that compute some isomorphism between  $\mathcal{A}$  and each computable copy of  $\mathcal{A}$ ; that is,  $\text{CatSpec}(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computably categorical}\} = \bigcap_{\mathcal{B} \cong \mathcal{A}} \text{IsoSpec}(\mathcal{A}, \mathcal{B})$ . We write  $\mathcal{D}$  for the set of all Turing degrees, and  $\mathcal{D}_{\geq}(\mathbf{d})$  for the set of Turing degrees that compute  $\mathbf{d}$ . If  $\text{CatSpec}(\mathcal{A}) = \mathcal{D}_{\geq}(\mathbf{d})$ , then  $\mathbf{d}$  is the *degree of categoricity* of  $\mathcal{A}$ .

The following is an example of a family of computable structures that are not computably categorical. We will use these particular structures in Section 3.3.

**Example 1.** Given a c.e. set  $X \subseteq \omega$  with a fixed computable enumeration  $\{x_i : i < \omega\}$ , define the ordering  $(\omega, <_X)$  where for all  $n < m < \omega$ , we have  $2n <_X 2m$  and  $2x_n <_X 2n + 1 <_X 2x_n + 2$ . It is not hard to check that  $(\omega, <_X)$  is a computable copy of  $(\omega, <)$ . Notice that  $k \in X$  if and only if there exists an  $n$  such that  $2k <_X n <_X 2k + 2$ , so the unique isomorphism  $f: (\omega, <) \cong (\omega, <_X)$  can be used to compute  $X$ . We can also computably build  $f$  using an oracle for  $X$ , so  $f \equiv_T X$ . That is,  $\text{IsoSpec}((\omega, <), (\omega, <_X)) = \mathcal{D}_{\geq}(\deg_T(X))$ .

In particular,  $\text{IsoSpec}((\omega, <), (\omega, <_{\mathbf{0}'})) = \mathcal{D}_{\geq}(\mathbf{0}')$ . Moreover, if  $(\omega, <)$  is any computable copy of  $(\omega, <)$ , the unique isomorphism from  $(\omega, <)$  to  $(\omega, <_{\mathbf{0}'})$  is  $\mathbf{0}'$ -computable, so  $\text{CatSpec}(\omega, <) = \mathcal{D}_{\geq}(\mathbf{0}')$ .

Let  $\mathcal{M}$  be any computable structure and let  $\{(\mathcal{A}_i, \mathcal{B}_i)\}_{i < \omega}$  be a list of all pairs of computable copies of  $\mathcal{M}$ . Then  $\text{CatSpec}(\mathcal{M}) = \bigcap_{i < \omega} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i)$ . For some computable structures  $\mathcal{M}$ , the categoricity spectrum is achieved by a finite intersection of isomorphism spectra. Bazhenov, Kalimullin & Yamaleev [3] defined the *spectral dimension* of  $\mathcal{M}$  as the least  $k \leq \omega$  such that there is a set  $X \subseteq \omega$  with  $|X| = k$  and  $\text{CatSpec}(\mathcal{M}) = \bigcap_{i \in X} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i)$ . We write  $\text{SpecDim}(\mathcal{M}) = k$  in this case.

If a structure has degree of categoricity  $\mathbf{d}$  and spectral dimension 1, we say that the structure has *strong degree of categoricity*  $\mathbf{d}$ . The study of strong degrees of categoricity motivated the introduction of spectral dimension. In the example above, we see that  $(\omega, <)$  has strong degree of categoricity  $\mathbf{0}'$ .

Fokina, Kalimullin & Miller [7] first introduced categoricity spectra and degrees of categoricity in 2010. A structure of Miller [9] (a particular algebraic field) was the first shown not to have a degree of categoricity. It has spectral dimension 1. Fokina, Frolov & Kalimullin [6] showed that for every non-zero c.e. degree  $\mathbf{d}$ , there is a  $\mathbf{d}$ -computably categorical rigid structure with no degree of categoricity. These structures have infinite spectral dimension. Bazhenov, Kalimullin & Yamaleev [3] and Csima & Stephenson [5] independently constructed rigid computable structures with degree of categoricity and finite spectral dimension greater than 1. These were the first examples of structures that have degrees of categoricity, but not strongly. Turetsky [13] constructed the first example of a computable structure with degree of categoricity and infinite spectral dimension.

It is still unknown whether there exists a computable structure  $\mathcal{M}$  with a degree of categoricity that is not the strong degree of categoricity of any structure. In Section 2.2 we expand on a result of Bazhenov, Kalimullin & Yamaleev [4] to obtain some properties that such a structure would have:

**Theorem 17** (Follows from Bazhenov, Kalimullin & Yamaleev, [2; 4]). If  $\mathbf{d}$  is the degree of categoricity of a computable structure  $\mathcal{M}$  and is not the strong degree of categoricity of any computable structure, then  $\mathcal{M}$  has infinite spectral dimension, an infinite automorphism group, and a pair of computable copies  $\mathcal{A}$  and  $\mathcal{B}$  such that  $\text{IsoSpec}(\mathcal{A}, \mathcal{B})$  is not *finitely generated* (that is, not equal to a finite union of cones).

Motivated by this, we are interested in studying isomorphism spectra that are not finitely generated. Section 2 introduces *computably composite structures*, a notion of effectively attaching a collection of computable structures to the points of another computable structure. We discuss the isomorphism spectra of computably composite structures in terms of the isomorphisms of their component structures. In Section 3, we use a specific class of computably composite structures to show that the class of isomorphism spectra is closed under computable unions:

**Theorem 29.** Given two uniformly computable collections of copies  $\mathbf{A} = \{\mathcal{A}_i : i < \omega\}$  and  $\mathbf{B} = \{\mathcal{B}_i : i < \omega\}$  such that for each  $i$ ,  $\mathcal{A}_i \cong \mathcal{B}_i$ , there exists a structure with two computable copies  $\mathcal{M} \cong \mathcal{N}$  such that  $\text{IsoSpec}(\mathcal{M}, \mathcal{N}) = \bigcup_{i < \omega} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i)$ .

In Section 3.3 we apply Theorem 29 to a result of Thomason [12] to construct a particular structure whose isomorphism spectrum is not finitely generated. Section 4 describes some connections to work of Harizanov, Miller & Morozov [8] on automorphism spectra of structures. Section 5 briefly considers the categoricity spectra of some computably composite structures and computes the categoricity spectrum of the particular structures from Section 3.3.

**Remark 2.** Every structure is uniformly effectively bi-interpretable with a graph (see Montalbán [10, Chapter 6.3]). Using techniques similar to the proof of [10, Lemma 6.3.8], it can be shown that effective bi-interpretability preserves isomorphism spectra and categoricity spectra so that it is sufficient to prove Theorems 17 and 29 in just the case of relational structures. Thus, in the following sections, we will assume that all structures are relational.

## 2 Computably composite structures

We begin by introducing the notion of *computably composite structures*. Informally, given a computable structure  $\mathcal{S}$ , we wish to attach a computable structure  $\mathcal{A}_x$  to each point  $x \in S$ . This new structure  $\mathcal{S}[\mathcal{A}_x : x \in S]$  will be computable. Its computable copies and the isomorphisms between those copies will have desirable properties.

Throughout this paper, we use calligraphic capital letters for copies of structures, and boldface capital letters for collections of copies of structures.

**Definition 3.** Let  $I$  be any computable set and let  $\mathbf{A} := \{\mathcal{A}_x : x \in I\}$  be a collection of computable copies of structures. We say that  $\mathbf{A}$  is *uniformly computable* if each  $\mathcal{A}_x$  is a particular computable copy of a computable  $\mathcal{L}_x$ -structure, the languages  $\{\mathcal{L}_x : x \in I\}$  are uniformly computable, and there is a computable function  $f : I \rightarrow \omega$  such that  $\varphi_{f(x)} = D(\mathcal{A}_x)$  for each  $x \in I$ . That is, the Turing machine with index  $f(x)$  computes the atomic diagram of the structure  $\mathcal{A}_x$ , see [1; 10].

**Remark 4.** In Definition 3, if  $x \neq y$ , then  $\mathcal{A}_x$  and  $\mathcal{A}_y$  may not be copies of the same computable structure, and may not even have the same language or universe. In particular, the universes of these copies may not be subsets of  $\omega$ . However, since they are uniformly computable, there are standard uniformly computable encodings of the universes into computable subsets of  $\omega$ .

**Definition 5** (Computably composite structures). Let  $\mathcal{S}$  be a computable  $\mathcal{L}_\mathcal{S}$ -structure with universe  $S$ , and let  $\mathbf{A} := \{\mathcal{A}_x : x \in S\}$  be a uniformly computable collection of copies such that the universes  $\{S\} \cup \{\mathcal{A}_x : x \in S\}$  are pairwise disjoint. We define the *computable composition of  $\mathcal{S}$  with  $\mathbf{A}$*  to be the structure  $\mathcal{S}[\mathbf{A}]$  with universe  $S \cup \bigcup_{x \in S} \mathcal{A}_x$ , relational language  $\mathcal{L} := \{\mu\} \cup \mathcal{L}_\mathcal{S} \cup \bigcup_{x \in S} \mathcal{L}_x$ , and satisfying:

- (1)  $\mu^{S[\mathbf{A}]} = \{(a, x) : x \in S, a \in \mathcal{A}_x\} \cup \{(x, x) : x \in S\}$ .
- (2) If  $R$  is an  $n$ -ary relation in  $\mathcal{L}_\mathcal{S}$  and  $x_0, \dots, x_{n-1} \in S$ , then  $R^{S[\mathbf{A}]}(x_0, \dots, x_{n-1})$  iff  $R^S(x_0, \dots, x_{n-1})$ .
- (3) If  $Q$  is an  $n$ -ary relation in  $\mathcal{L}_x$  and  $a_0, \dots, a_{n-1} \in \mathcal{A}_x$ , then  $Q^{S[\mathbf{A}]}(a_0, \dots, a_{n-1})$  iff  $Q^{\mathcal{A}_x}(a_0, \dots, a_{n-1})$ .
- (4) No other relations hold.

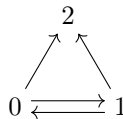
A structure of the form  $\mathcal{S}[\mathbf{A}]$  is called a *computably composite structure*. The structure  $\mathcal{S}$  is the *base structure* of  $\mathcal{S}[\mathbf{A}]$ , and the structures  $\{\mathcal{A}_x : x \in S\}$  are the *component structures*.

**Remark 6.** In a computably composite structure,  $\mu$  is a directed edge. Every element has exactly one outward  $\mu$ -edge, so we may use the notation  $\mu(z)$  to denote the unique element for which  $(z, \mu(z)) \in \mu$ . This is computable since the element is unique and the universe of the structure is computable. Also notice that  $\mu(z) \in S$  for all elements  $z$  of  $\mathcal{S}[\mathbf{A}]$ . We may think of  $\mu$  as connecting members of  $\mathcal{A}_x$  to the point  $x \in S$ .

**Remark 7.** In  $\mathcal{S}$  and in each component structure, the relations are uniformly computable. By the uniform computability of  $\mathbf{A}$ , the relations of  $\mathcal{S}[\mathbf{A}]$  are also uniformly computable. Since  $\{S\} \cup \{\mathcal{A}_x : x \in S\}$  are pairwise disjoint and uniformly computable, the relation  $\mu^{S[\mathbf{A}]}$  is computable. Thus,  $\mathcal{S}[\mathbf{A}]$  is a computable structure.

**Remark 8.**  $\mathcal{S}$  is a reduct of the substructure of  $\mathcal{S}[\mathbf{A}]$  with universe  $S$  and for each  $x \in S$ ,  $\mathcal{A}_x$  is a reduct of the substructure of  $\mathcal{S}[\mathbf{A}]$  with universe  $\mathcal{A}_x$ . As these reducts remove only trivial relations, we will identify them with the actual substructures.

**Example 9.** We consider a simple illustrative example where  $\mathcal{S} = (\{0, 1, 2\}, E)$  is the finite directed graph where  $E = \{(0, 1), (1, 0), (0, 2), (1, 2)\}$ .



Now take  $\mathbf{A} := \{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2\}$  where  $\mathcal{A}_0 := (\{0\} \times \omega, <)$ ,  $\mathcal{A}_1 := (\{1\} \times \omega, <)$ ,  $\mathcal{A}_2 := (\{2\} \times \mathbb{Z}, <)$ , and where  $<$  is interpreted in all structures as the usual ordering in the second coordinate. We have relabeled  $\omega$ ,  $\omega$ , and  $\mathbb{Z}$  to  $\{0\} \times \omega$ ,  $\{1\} \times \omega$  and  $\{2\} \times \mathbb{Z}$  respectively so that the universes are disjoint. Then the computably composite structure  $\mathcal{S}[\mathbf{A}]$  has universe  $\{0, 1, 2\} \cup (\{0\} \times \omega) \cup (\{1\} \times \omega) \cup (\{2\} \times \mathbb{Z})$ , and language  $(\mu, E, <)$ .

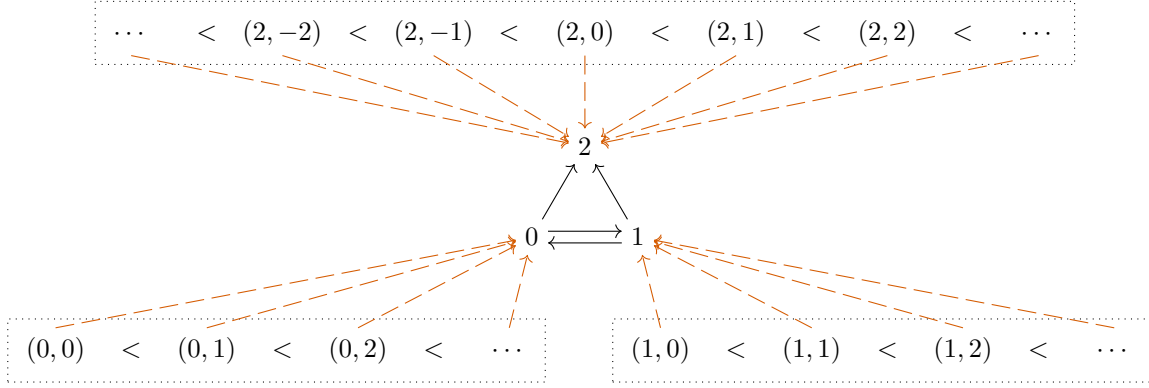


Figure 1: The black arrows represent the directed edge relation  $E$  and the dashed orange arrows represent the directed edge relation  $\mu$ . The dotted boxes indicate the subsets that generate substructures  $\mathcal{A}_0$ ,  $\mathcal{A}_1$ , and  $\mathcal{A}_2$ . The  $\mu$ -self-loops on 0, 1, 2 are omitted.

We briefly consider the automorphisms of this structure. Every automorphism of  $\mathcal{S}[\mathbf{A}]$  fixes 2, and may or may not swap 0 and 1. If an automorphism swaps 0 and 1, it also must swap each  $(0, n)$  and  $(1, n)$ . An automorphism may shift each  $(2, n)$  to  $(2, n + k)$  for some  $k \in \mathbb{Z}$ . The automorphism is determined after choosing whether or not to swap 0 with 1 and choosing a  $k \in \mathbb{Z}$ , so automorphisms of  $\mathcal{S}[\mathbf{A}]$  are exactly the maps of the form  $\varrho = \vartheta \cup \psi_0 \cup \psi_1 \cup \psi_2$  where  $\vartheta \in \text{Aut}(\mathcal{S})$ , and  $\psi_i: \mathcal{A}_i \cong \mathcal{A}_{\vartheta(i)}$  for  $i \in \{0, 1, 2\}$ . We will show in the next section that this characterization extends to isomorphisms between arbitrary pairs of computably composite structures.

## 2.1 Isomorphisms between computably composite structures

First, we will show that every computable copy of a computably composite structure is also computably composite.

**Proposition 10.** Suppose  $\mathcal{M}$  is a computable copy of  $\mathcal{S}[\mathbf{A}]$  via isomorphism  $\varrho: \mathcal{S}[\mathbf{A}] \cong \mathcal{M}$ . Let  $\mathcal{G}$  be the substructure of  $\mathcal{M}$  generated by  $\varrho(S)$ . For each  $g \in G$  let  $\mathcal{B}_g$  be the substructure of  $\mathcal{M}$  generated by  $\varrho(A_{\varrho^{-1}(g)})$  and let  $\mathbf{B} := \{\mathcal{B}_g : g \in G\}$ . Then  $\mathcal{G}[\mathbf{B}] = \mathcal{M}$  is a computably composite structure.

*Proof.* Since  $\{S\} \cup \{A_x : x \in S\}$  are pairwise disjoint and  $\varrho$  is a bijection, it follows that  $\{G\} \cup \{\mathcal{B}_g : g \in G\}$  are also pairwise disjoint. It is clear that the construction of  $\mathcal{G}[\mathbf{B}]$  yields exactly the structure  $\mathcal{M}$ . The substructure  $\mathcal{G}$  of  $\mathcal{M}$  is generated by the set  $G = \{x \in M : \mu^{\mathcal{M}}(x, x)\}$ , which is computable, so  $\mathcal{G}$  is computable. It remains to verify that  $\mathbf{B}$  is uniformly computable.

For each  $g \in G$ ,  $\mathcal{B}_g$  is the substructure of  $\mathcal{M}$  generated by the set  $B_g = \{b \in M : \mu^{\mathcal{M}}(b, g) \wedge b \neq g\}$ , which is uniformly computable in  $g$ . Then  $\mathbf{B} = \{\mathcal{B}_g\}_{g \in G}$  is uniformly computable, since a relation holds over elements of  $B_g$  if and only if the relation holds over the same elements of  $\mathcal{M}$ .  $\square$

Next, we characterize the isomorphisms between two computably composite structures  $\mathcal{S}[\mathbf{A}]$  and  $\mathcal{G}[\mathbf{B}]$ . Informally, these are exactly the maps that induce an isomorphism from  $\mathcal{S}$  to  $\mathcal{G}$  and induce isomorphisms between the corresponding component structures.

**Proposition 11.** Suppose  $\mathcal{S}[\mathbf{A}]$  and  $\mathcal{G}[\mathbf{B}]$  are isomorphic computably composite structures. Then the isomorphisms from  $\mathcal{S}[\mathbf{A}]$  to  $\mathcal{G}[\mathbf{B}]$  are exactly the maps of the form  $\varrho = \vartheta \cup \bigcup_{x \in S} \psi_x$  where  $\vartheta: \mathcal{S} \cong \mathcal{G}$  and  $\psi_x: \mathcal{A}_x \cong \mathcal{B}_{\vartheta(x)}$  for each  $x \in S$ .

*Proof.* For any such  $\vartheta$  and  $\{\psi_x\}_{x \in S}$ , since  $\text{dom}(\vartheta) \cup \bigcup_{x \in S} \text{dom}(\psi_x) = S \cup \bigcup_{x \in S} A_x$ , the map  $\varrho$  is actually a function from the universe of  $\mathcal{S}[\mathbf{A}]$  to the universe of  $\mathcal{G}[\mathbf{B}]$ . Since  $\vartheta(S) \cup \bigcup_{x \in S} \psi_x(A_x) = G \cup \bigcup_{x \in S} B_{\vartheta(x)}$  and  $\vartheta$  is a bijection,  $\varrho$  is a bijection. It is easily checked that all relations are preserved by  $\varrho$ , so every such  $\varrho$  is an isomorphism.

Let  $\varrho: \mathcal{S}[\mathbf{A}] \rightarrow \mathcal{G}[\mathbf{B}]$  be any isomorphism. We have  $x \in S \Leftrightarrow \mu^{\mathcal{S}[\mathbf{A}]}(x, x)$  and  $g \in G \Leftrightarrow \mu^{\mathcal{G}[\mathbf{B}]}(g, g)$ , so, since  $\varrho$  is an isomorphism, we must have that  $\vartheta := \varrho|_S$  is an isomorphism from  $\mathcal{S}$  to  $\mathcal{G}$ . For any  $x \in S$ ,  $a \in A_x \Leftrightarrow (\mu^{\mathcal{S}[\mathbf{A}]}(a, x) \wedge \neg \mu^{\mathcal{S}[\mathbf{A}]}(a, a))$  and

$$g \in B_{\vartheta(x)} \Leftrightarrow \left( \mu^{\mathcal{G}[\mathbf{B}]}(g, \vartheta(x)) \wedge \neg \mu^{\mathcal{G}[\mathbf{B}]}(g, g) \right) \Leftrightarrow \left( \mu^{\mathcal{G}[\mathbf{B}]}(g, \varrho(x)) \wedge \neg \mu^{\mathcal{G}[\mathbf{B}]}(g, g) \right),$$

so, since  $\varrho$  is an isomorphism, we must have that  $\psi_x := \varrho|_{A_x}$  is an isomorphism from  $\mathcal{A}_x$  to  $\mathcal{B}_{\vartheta(x)}$ . As this covers the domain of  $\mathcal{S}[\mathbf{A}]$ , we must have  $\varrho = \vartheta \cup \bigcup_{x \in S} \psi_x$ .  $\square$

**Remark 12.** An important case is when the base structure  $\mathcal{S}$  is *computably stable*: for every computable copy  $\mathcal{G}$  of  $\mathcal{S}$ , every isomorphism from  $\mathcal{S}$  to  $\mathcal{G}$  is computable. When  $\mathcal{S}$  is computably stable and  $\mathcal{S}[\mathbf{A}]$  is computably composite, Proposition 10 implies that, up to computable isomorphism, the computable copies of  $\mathcal{S}[\mathbf{A}]$  are exactly the computably composite structures  $\mathcal{S}[\mathbf{B}]$  where  $\mathcal{A}_x \cong \mathcal{B}_x$  for all  $x \in S$ . Proposition 11 implies that the complexity of the isomorphisms between  $\mathcal{S}[\mathbf{A}]$  and  $\mathcal{S}[\mathbf{B}]$  depend only on the complexity of the isomorphisms between the corresponding component structures. This will be an important property of the structure defined in Section 3.

## 2.2 Degrees of categoricity that are never strong

It is not known whether there exists a degree of categoricity that is not the strong degree of categoricity of any computable structure. We adapt a result of Bazhenov, Kalimullin, and Yamaleev which serves as motivation for the main result of this paper. For this, we define a particularly simple class of computably composite structures:

**Example 13.** For each  $n < \omega$ , define the directed graph  $\mathcal{P}_n = (\{0, \dots, n-1\}, E_n)$  where we define  $E_n = \{(k, k+1) : k < n-1\}$ . If  $\{\mathcal{A}_i : i < n\}$  is any collection of computable structures, it is uniformly computable since it is finite, so  $\mathcal{P}_n[\{i\} \times \mathcal{A}_i : i < n]$  is a computably composite structure (we relabel the universe  $A_i$  as  $\{i\} \times A_i$  to ensure disjointness). We may think of  $\mathcal{P}_n[\{i\} \times \mathcal{A}_i : i < n]$  as a *directed path of the structures*:  $\mathcal{A}_0 \rightarrow \mathcal{A}_1 \rightarrow \dots \rightarrow \mathcal{A}_{n-1}$ . Since  $\mathcal{P}_n$  is computably stable, it follows from Propositions 10 and 11 that the computable copies of  $\mathcal{P}_n[\{i\} \times \mathcal{A}_i : i < n]$  are (up to computable isomorphism) of the form  $\mathcal{P}_n[\{i\} \times \mathcal{B}_i : i < n]$  where  $\mathcal{B}_i$  is a computable copy of  $\mathcal{A}_i$  for each  $i < n$ . Each isomorphism between these paths is Turing-equivalent to a union  $\bigcup_{i < n} \psi_i$ , where  $\psi_i: \mathcal{A}_i \cong \mathcal{B}_i$  for each  $i < n$ . Thus, their isomorphism spectrum is exactly  $\bigcap_{i < n} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i)$ .

**Theorem 14** (Follows from Bazhenov, Kalimullin & Yamaleev, [4, Proposition 1b]). If  $\mathcal{M}$  is a computable structure with spectral dimension  $n < \omega$  and degree of categoricity  $\mathbf{d}$ , then  $\mathcal{P}_n[\{i\} \times \mathcal{M} : i < n]$  has strong degree of categoricity  $\mathbf{d}$ .

*Proof.* Since  $\mathcal{M}$  has spectral dimension  $n$ , there are pairs  $(\mathcal{A}_i, \mathcal{B}_i)_{i < n}$  of computable copies of  $\mathcal{M}$  such that  $\bigcap_{i < n} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i) = \text{CatSpec}(\mathcal{M}) = \mathcal{D}_{\geq}(\mathbf{d})$ . Now  $\mathcal{P}_n[\mathcal{A}_i : i < n]$  and  $\mathcal{P}_n[\mathcal{B}_i : i < n]$  are computably composite structures, as in the previous example, and are both isomorphic to  $\mathcal{P}_n[\{i\} \times \mathcal{M} : i < n]$ . Thus,  $\text{IsoSpec}(\mathcal{P}_n[\mathcal{A}_i : i < n], \mathcal{P}_n[\mathcal{B}_i : i < n]) = \bigcap_{i < n} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i) = \mathcal{D}_{\geq}(\mathbf{d})$ . Since every computable copy of  $\mathcal{P}_n[\{i\} \times \mathcal{M} : i < n]$  is a path  $\mathcal{N}_0 \rightarrow \dots \rightarrow \mathcal{N}_{n-1}$  with each  $\mathcal{N}_i$  a computable copy of  $\mathcal{M}$ , and since  $\mathcal{M}$  is  $\mathbf{d}$ -computably categorical, every isomorphism between these paths is computable in  $\mathbf{d}$ . So  $\mathcal{P}_n[\mathcal{M}]$  has degree of categoricity  $\mathbf{d}$  and spectral dimension 1 witnessed by copies  $\mathcal{P}_n[\mathcal{A}_i : i < n]$  and  $\mathcal{P}_n[\mathcal{B}_i : i < n]$ .  $\square$

**Definition 15.** An upward-closed set of Turing degrees  $\mathcal{Z} \subseteq \mathcal{D}$  is *generated* by a subset  $\mathcal{X} \subseteq \mathcal{Z}$  if  $\mathcal{Z} = \bigcup_{\mathbf{d} \in \mathcal{X}} \mathcal{D}_{\geq}(\mathbf{d})$ . We say that  $\mathcal{Z}$  is *finitely generated* if it is generated by a finite set of Turing degrees. That is, if it is equal to a union of finitely many cones.

Bazhenov, Kalimullin & Yamaleev [2] show that any computable structure with a degree of categoricity and infinite spectral dimension must have a pair of copies whose isomorphism spectrum is not finitely generated.

**Theorem 16** (Bazhenov, Kalimullin & Yamaleev, [2, Theorem 3.1]). Let  $\mathcal{M}$  be a computable structure with degree of categoricity  $\mathbf{d}$ , and infinite spectral dimension. Then there are computable copies  $\mathcal{A}$  and  $\mathcal{B}$  of  $\mathcal{M}$  such that  $\text{IsoSpec}(\mathcal{A}, \mathcal{B})$  is not finitely generated.

By combining these results, we obtain the following necessary conditions for any never-strong degree of categoricity:

**Theorem 17** (Follows from Bazhenov, Kalimullin & Yamaleev, [2; 4]). If  $\mathbf{d}$  is the degree of categoricity of a computable structure  $\mathcal{M}$  and is not the strong degree of categoricity of any computable structure, then  $\mathcal{M}$  has infinite spectral dimension, an infinite automorphism group, and a pair of computable copies  $\mathcal{A}$  and  $\mathcal{B}$  such that  $\text{IsoSpec}(\mathcal{A}, \mathcal{B})$  is not finitely generated.

Motivated by this result, we now work toward constructing pairs of computably composite structures whose isomorphism spectra are not finitely generated.

### 3 An infinite base structure

In the proof of Theorem 14, we used computably composite structures on the finite base structure  $\mathcal{P}_n$  to construct new computable structures whose isomorphism spectra are the intersection of the original  $n$  isomorphism spectra. Similar constructions yield finite unions of isomorphism spectra (for example, if  $\mathcal{A}_1 \cong \mathcal{B}_1$  and  $\mathcal{A}_2 \cong \mathcal{B}_2$ , then undirected paths  $\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{B}_2 - \mathcal{B}_1$  and  $\mathcal{A}_1 - \mathcal{B}_2 - \mathcal{A}_2 - \mathcal{B}_1$  have isomorphism spectrum  $\text{IsoSpec}(\mathcal{A}_1, \mathcal{B}_1) \cup \text{IsoSpec}(\mathcal{A}_2, \mathcal{B}_2)$ ). In order to obtain examples of isomorphism spectra, we will show in this section that we can form arbitrary *infinite* unions of isomorphism spectra. To do this, we define an infinite base structure,  $\mathcal{H}$ , to replace the paths in the previous examples, and show that computably composite structures on  $\mathcal{H}$  are witnesses to the existence of the isomorphism spectra required for Theorem 29.

**Definition 18.** We define the structure  $\mathcal{H} = (H, \{D_i\}_{i < \omega}, \{E_i\}_{i < \omega})$  as follows. Define  $H := [\omega]^{<\omega} \cup (\omega \times \{0, 1\})$ , so an element of  $H$  is either a finite subset of  $\omega$ , or a pair  $(i, a)$  where  $i < \omega$  and  $a < 2$ . By identifying finite sets with their characteristic functions, we may think of  $[\omega]^{<\omega}$  as the collection of infinite binary strings with finitely many 1's. We will also think of  $[\omega]^{<\omega}$  as the vertices of an infinite-dimensional hypercube in which vertices  $X, Y \in [\omega]^{<\omega}$  are adjacent if  $|X \Delta Y| = 1$ .

The structure  $\mathcal{H}$  is equipped with undirected edge relations  $\{E_i\}_{i < \omega}$  forming the edges of the hypercube running along each dimension. That is, for any  $X, Y \in [\omega]^{<\omega}$  and  $i < \omega$ ,  $E_i(X, Y) \Leftrightarrow E_i(Y, X) \Leftrightarrow X \Delta Y = \{i\}$ . Figure 2 is an illustration of the 3-dimensional substructure of the hypercube using only the subsets of  $\{0, 1, 2\}$  and the edge relations  $E_0, E_1$ , and  $E_2$  between them.

Notice that for any  $X \in [\omega]^{<\omega}$  and any  $i < \omega$ ,  $X \Delta \{i\}$  is the unique element of  $[\omega]^{<\omega}$  with the property  $E_i(X, X \Delta \{i\})$ . For each  $i \in \omega$ , we will think of the tuples  $(i, 0), (i, 1) \in H$  as labeling the opposing hyperfaces  $\{X \in [\omega]^{<\omega} : i \notin X\}$  and  $\{X \in [\omega]^{<\omega} : i \in X\}$  of the hypercube as illustrated in Figure 3. The structure  $\mathcal{H}$  is also equipped with  $\omega$ -many directed edge relations defined so that for any  $X \in [\omega]^{<\omega}$ , any  $i < \omega$ , and any  $a < 2$ ,  $D_i(X, (i, a))$  iff  $X(i) = a$ . That is, for each finite set  $X$  and each  $i < \omega$ , there is a directed  $D_i$ -edge leaving  $X$ . If  $i \notin X$ , then that edge goes to  $(i, 0)$ , and if  $i \in X$ , then that edge goes to  $(i, 1)$ . Figure 3 illustrates  $D_0, D_1$ , and  $D_2$ .

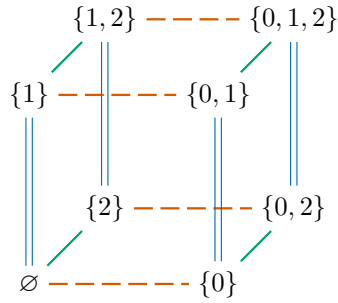


Figure 2: Subsets of  $\{0, 1, 2\}$  and edge relations  $E_0$  (orange dashed),  $E_1$  (blue double), and  $E_2$  (green).

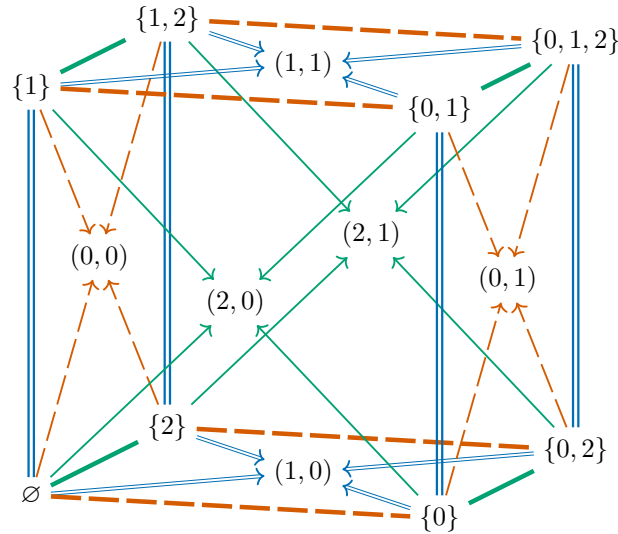


Figure 3: Subsets of  $\{0, 1, 2\}$ , edge relations  $E_0$  (orange dashed lines),  $E_1$  (blue double lines), and  $E_2$  (green lines), hyperfaces  $\{0, 1, 2\} \times \{0, 1\}$ , and edge relations  $D_0$  (orange dashed arrows),  $D_1$  (blue double arrows), and  $D_2$  (green arrows).

Using a computable bijection from  $\omega$  to  $[\omega]^{<\omega}$ , we see that  $H$  is computable and that  $\{E_i\}_{i<\omega} \cup \{D_i\}_{i<\omega}$  is uniformly computable, so  $\mathcal{H}$  is a computable structure.

**Definition 19.** For each  $X \in [\omega]^{<\omega}$ , we will also define a map  $h_X: H \rightarrow H$  that is uniformly computable in  $X$ . For  $Y \in [\omega]^{<\omega}$ , let  $h_X(Y) := X \triangle Y$ , and for  $(i, a) \in \omega \times \{0, 1\}$ , let

$$h_X(i, a) := (i, a + X(i)) = \begin{cases} (i, a) & \text{if } i \notin X, \\ (i, 1 - a) & \text{if } i \in X. \end{cases}$$

**Remark 20.** Notice that  $h_\emptyset$  is the identity on  $H$ . Fix  $i < \omega$ .

- (1) If  $i \notin X$ , then  $h_{\{i\}}(X) = X \cup \{i\}$  and  $h_{\{i\}}(X \cup \{i\}) = X$ .
- (2)  $h_{\{i\}}(i, 0) = (i, 1)$  and  $h_{\{i\}}(i, 1) = (i, 0)$ .
- (3) If  $j \neq i$ ,  $h_{\{i\}}(j, 0) = (j, 0)$  and  $h_{\{i\}}(j, 1) = (j, 1)$ .

This shows that  $h_{\{i\}}$  is the reflection of the hypercube along the  $i^{\text{th}}$  dimension.

### 3.1 Automorphisms of the base structure

By Proposition 11, isomorphisms between copies of a computably composite structure on  $\mathcal{H}$  depend partly on the isomorphisms between computable copies of  $\mathcal{H}$ . We will see that it is sufficient to consider the automorphisms of  $\mathcal{H}$ .

**Lemma 21.** Let  $\vartheta$  be an automorphism of  $\mathcal{H}$ . If  $X \in [\omega]^{<\omega}$ , then  $\vartheta(X) \in [\omega]^{<\omega}$ . If  $(i, a) \in \omega \times \{0, 1\}$ , then either  $\vartheta(i, a) = (i, 0)$  or  $\vartheta(i, a) = (i, 1)$ .

*Proof.* We consider the number of inward and outward  $D_i$  edges at each point in  $H$ :

- (1) If  $X \in [\omega]^{<\omega}$ , then  $X$  has exactly one outward directed  $D_i$ -edge for each  $i < \omega$ .
- (2) If  $(i, a) \in \omega \times \{0, 1\}$ , then  $(i, a)$  has no outward directed  $D_j$ -edges for any  $j < \omega$ .
- (3) If  $(i, a) \in \omega \times \{0, 1\}$ , then  $(i, a)$  has infinitely many incoming  $D_i$ -edges.
- (4) If  $(i, a) \in \omega \times \{0, 1\}$ , then  $(i, a)$  has no incoming  $D_j$ -edges when  $j \neq i$ .

As these properties must be preserved by any automorphism, the result follows.  $\square$

**Lemma 22.** For  $X, Y \in [\omega]^{<\omega}$ ,  $h_X \circ h_Y = h_{X \triangle Y}$ . In particular,  $h_X \circ h_X = h_\emptyset = id_H$  and  $h_X \circ h_{\{i\}} = h_{X \cup \{i\}}$  when  $i \notin X$ .

*Proof.* For  $Z \in [\omega]^{<\omega}$ , we have  $h_X \circ h_Y(Z) = X \triangle (Y \triangle Z) = (X \triangle Y) \triangle Z = h_{X \triangle Y}(Z)$ . For  $(i, a) \in \omega \times \{0, 1\}$ , we have  $h_X \circ h_Y(i, a) = h_X(i, a + Y(i)) = (i, a + Y(i) + X(i)) = (i, a + (X \triangle Y)(i)) = h_{X \triangle Y}(i, a)$ .  $\square$

**Proposition 23.** For each  $X \in [\omega]^{<\omega}$ , the map  $h_X$  is an automorphism of  $\mathcal{H}$ .

*Proof.* By Lemma 22,  $h_X \circ h_X = id_H$ , so  $h_X$  is bijective. If  $i \notin X$ , then  $i \in Y \Leftrightarrow i \in (X \triangle Y)$ . If  $i \in X$ , then  $i \in Y \Leftrightarrow i \notin (X \triangle Y)$ . So in general, we have  $Y(i) = a \Leftrightarrow (X \triangle Y)(i) = a + X(i)$ . The following equivalences show that  $h_X$  preserves all relations  $E_i$  and  $D_i$  in  $\mathcal{H}$  and so is an automorphism:

$$\begin{aligned}
 D_i(Y, (i, a)) &\Leftrightarrow Y(i) = a \\
 &\Leftrightarrow (X \triangle Y)(i) = a + X(i) \\
 &\Leftrightarrow h_X(Y)(i) = a + X(i) \\
 &\Leftrightarrow D_i(h_X(Y), (i, a + X(i))) \\
 &\Leftrightarrow D_i(h_X(Y), h_X(i, a)), \\
 E_i(Y, Z) &\Leftrightarrow Y \triangle Z = \{i\} \\
 &\Leftrightarrow (Y \triangle X) \triangle (Z \triangle X) = \{i\} \\
 &\Leftrightarrow E_i(Y \triangle X, Z \triangle X) \\
 &\Leftrightarrow E_i(h_X(Y), h_X(Z)).
 \end{aligned}$$

$\square$

**Proposition 24.** For each  $X \in [\omega]^{<\omega}$ , the map  $h_X$  is the unique automorphism sending  $\emptyset$  to  $X$ .

*Proof.* Every  $h_X$  is an automorphism by Proposition 23. Suppose  $\vartheta: \mathcal{H} \cong \mathcal{H}$  is an automorphism and that  $\vartheta(\emptyset) = X$ . By Lemma 21, for each  $i < \omega$  there is an  $a_i \in \{0, 1\}$  such that  $\vartheta(i, 0) = (i, a_i)$  and  $\vartheta(i, 1) = (i, 1 + a_i)$ . Since  $(\emptyset, (i, 0)) \in D_i$  and  $\vartheta$  is an automorphism, we have  $(\vartheta(\emptyset), \vartheta(i, 0)) = (X, (i, a_i)) \in D_i$  for each  $i < \omega$ . But then, by the definition of  $D_i$ ,  $a_i = X(i)$  for all  $i$ , so  $\vartheta(i, 0) = (i, a_i) = (i, X(i)) = h_X(i, 0)$  and  $\vartheta(i, 1) = (i, 1 + a_i) = (i, 1 + X(i)) = h_X(i, 1)$ . Thus,  $\vartheta(i, a) = h_X(i, a)$  for all  $(i, a) \in \omega \times \{0, 1\}$ .

Let  $Y \in [\omega]^{<\omega}$ , so by Lemma 21,  $Z := \vartheta(Y) \in [\omega]^{<\omega}$ . Since  $\vartheta(i, a) = h_X(i, a) = (i, a + X(i))$ , we have  $Y(i) = a \Leftrightarrow D_i(Y, (i, a)) \Leftrightarrow D_i(Z, (i, a + X(i))) \Leftrightarrow Z(i) = a + X(i)$ , so  $Z(i) = Y(i) + X(i)$  for all  $i < \omega$ , and so  $\vartheta(Y) = Z = X \triangle Y = h_X(Y)$ . Since  $Y \in [\omega]^{<\omega}$  is arbitrary, we have  $\vartheta = h_X$ .  $\square$

**Corollary 25.** The set of automorphisms of  $\mathcal{H}$  is  $\{h_X : X \in [\omega]^{<\omega}\}$ .

*Proof.* By Lemma 21, every automorphism of  $\mathcal{H}$  sends  $\emptyset$  to some  $X \in [\omega]^{<\omega}$ , so by Proposition 24 is equal to  $h_X$ . The result then follows from Proposition 23.  $\square$

**Remark 26.** By Lemma 22, we have that if  $X = \{x_1, x_2, \dots, x_k\} \in [\omega]^{<\omega}$ , then  $h_X = h_{\{x_1\}} \circ h_{\{x_2\}} \circ \dots \circ h_{\{x_k\}}$ . Since  $h_{\{i\}}$  is the reflection of the hypercube along the  $i^{\text{th}}$  dimension, each  $h_X$  is a finite composition of reflections along the dimensions corresponding to the elements of  $X$ .

From Remark 12, the following proposition means that if  $\mathcal{H}[\mathbf{A}]$  is a computably composite structure on  $\mathcal{H}$ , the computable copies of  $\mathcal{H}[\mathbf{A}]$  are up to computable isomorphism exactly the computably composite structures of the form  $\mathcal{H}[\mathbf{B}]$  where  $\mathcal{B}_z \cong \mathcal{A}_z$  for all  $z \in H$ . Hence, the complexity of an isomorphism between two computably composite structures on  $\mathcal{H}$  depends only on the isomorphisms between the corresponding component structures.

**Proposition 27.** The structure  $\mathcal{H}$  is computably stable. That is, if  $\mathcal{G}$  is a computable copy of  $\mathcal{H}$ , then every isomorphism from  $\mathcal{H}$  to  $\mathcal{G}$  is computable.

*Proof.* Let  $f$  be any isomorphism from  $\mathcal{H}$  to  $\mathcal{G}$ . We may assume that  $\mathcal{G}$  has universe  $\omega$ . Define  $n_\emptyset := f(\emptyset)$ . By Proposition 24, the automorphisms of  $\mathcal{H}$  are determined by where  $\emptyset$  is mapped to. Thus,  $f$  is determined by  $f(\emptyset)$ . We now computably build  $f$  out from  $\emptyset$ .

Suppose we know  $f(X) \in \omega$  for some  $X \in [\omega]^{<\omega}$  and suppose  $n \notin X$ . Then  $X \cup \{n\}$  is the unique element of  $[\omega]^{<\omega}$  with the property  $E_n^{\mathcal{H}}(X, X \cup \{n\})$ . Thus,  $f(X \cup \{n\})$  is the unique element of  $\omega$  with the property  $E_n^{\mathcal{G}}(f(X), f(X \cup \{n\}))$  and can be found by iterating through  $\omega$  in finitely many steps since  $E_n^{\mathcal{G}}$  is computable. Since we know  $f(\emptyset)$ , we can find  $f(Z)$  for any  $Z \in [\omega]^{<\omega}$  by iterating this process once for each of the (finitely many) elements of  $Z$ .

Fix  $i \in \omega$ . Then  $(i, 0)$  is the unique element of  $H$  with the property  $D_i^{\mathcal{H}}(\emptyset, (i, 0))$ . Thus,  $f(i, 0)$  is the unique element of  $\omega$  with the property  $D_i^{\mathcal{G}}(f(\emptyset), f(i, 0))$ . Since we know  $f(\emptyset)$ ,  $f(i, 0)$  can be found by iterating through  $\omega$  in finitely many steps. Similarly,  $f(i, 1)$  is the unique element of  $\omega$  with the property  $D_i^{\mathcal{G}}(\{i\}, (i, 1))$ . Since we can computably find  $f(\{i\})$ , we can find  $f(i, 1)$  by iterating through  $\omega$  in finitely many steps.  $\square$

**Corollary 28.** The structure  $\mathcal{H}$  is computably categorical.

## 3.2 Computable unions of isomorphism spectra

We now show that the class of isomorphism spectra is closed under computable unions:

**Theorem 29.** Given two uniformly computable collections of copies  $\mathbf{A} = \{\mathcal{A}_i : i < \omega\}$  and  $\mathbf{B} = \{\mathcal{B}_i : i < \omega\}$  such that for each  $i$ ,  $\mathcal{A}_i \cong \mathcal{B}_i$ , there exists a structure with two computable copies  $\mathcal{M} \cong \mathcal{N}$  such that  $\text{IsoSpec}(\mathcal{M}, \mathcal{N}) = \bigcup_{i < \omega} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i)$ .

*Proof.* Fix two arbitrary uniformly computable collections of copies  $\mathbf{A} = \{\mathcal{A}_i : i < \omega\}$  and  $\mathbf{B} = \{\mathcal{B}_i : i < \omega\}$  with arbitrary uniformly computable relational languages  $\{\mathcal{L}_i : i < \omega\}$  such that for each  $i$ ,  $\mathcal{A}_i \cong \mathcal{B}_i$ . We will define  $\mathcal{M}$  and  $\mathcal{N}$ , two computably composite structures on  $\mathcal{H}$ , and then verify that they satisfy the statement of Theorem 29.

**Definition 30.** We first define uniformly computable collections of copies  $\mathbf{M} = \{\mathcal{M}_z : z \in H\}$  and  $\mathbf{N} = \{\mathcal{N}_z : z \in H\}$ . Each  $\mathcal{M}_z$  and each  $\mathcal{N}_z$  will be of the form  $\{z\} \times \mathcal{A}_k$  or  $\{z\} \times \mathcal{B}_k$  for some  $k$ , ensuring that the universes are disjoint (see Figure 4):

- (1) For  $i \in \omega$ , define  $\mathcal{M}_{(i,0)} = \mathcal{N}_{(i,0)} := \{(i, 0)\} \times \mathcal{A}_{i+1}$ .
- (2) For  $i \in \omega$ , define  $\mathcal{M}_{(i,1)} = \mathcal{N}_{(i,1)} := \{(i, 1)\} \times \mathcal{B}_{i+1}$ .
- (3) For  $X \in [\omega]^{<\omega}$ , define  $\mathcal{M}_X := \begin{cases} \{X\} \times \mathcal{A}_0 & \text{if } |X| \text{ is even,} \\ \{X\} \times \mathcal{B}_0 & \text{if } |X| \text{ is odd.} \end{cases}$
- (4) For  $X \in [\omega]^{<\omega}$ , define  $\mathcal{N}_X := \begin{cases} \{X\} \times \mathcal{B}_0 & \text{if } |X| \text{ is even,} \\ \{X\} \times \mathcal{A}_0 & \text{if } |X| \text{ is odd.} \end{cases}$

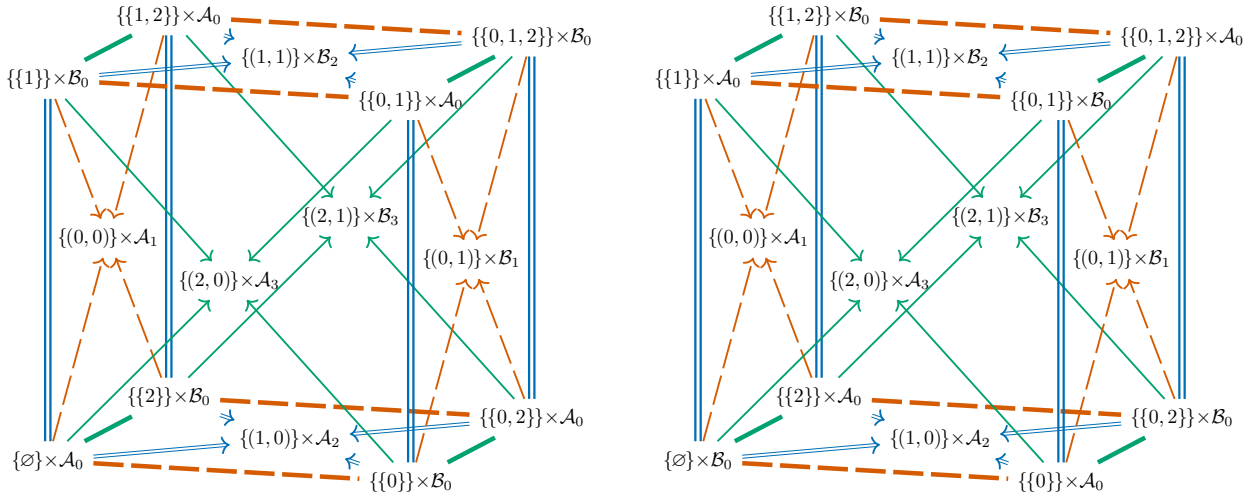


Figure 4: The first three dimensions of  $\mathcal{M} = \mathcal{H}[\mathbf{M}]$  (left) and  $\mathcal{N} = \mathcal{H}[\mathbf{N}]$  (right).

For  $X, Y \in [\omega]^{<\omega}$ , define functions  $\alpha_{X,Y}$  and  $\beta_{X,Y}$  so that for  $a \in A_0$ ,  $\alpha_{X,Y}(X, a) := (Y, a)$ , and for  $b \in B_0$ ,  $\beta_{X,Y}(X, b) := (Y, b)$ . Notice that  $\alpha_{X,Y}$  is a computable isomorphism from  $\{X\} \times A_0$  to  $\{Y\} \times A_0$  and  $\beta_{X,Y}$  is a computable isomorphism from  $\{X\} \times B_0$  to  $\{Y\} \times B_0$ . Thus, every copy in  $\{\mathcal{M}_X, \mathcal{N}_X : X \in [\omega]^{<\omega}\}$  is computably isomorphic to either  $\mathcal{M}_\emptyset$  via  $\alpha_{X,\emptyset}$  or to  $\mathcal{N}_\emptyset$  via  $\beta_{X,\emptyset}$ . The copies  $\mathcal{M}_\emptyset$  and  $\mathcal{N}_\emptyset$  are isomorphic, but possibly not computably so. Moreover,  $\alpha_{X,Y}$  and  $\beta_{X,Y}$  are uniformly computable in  $X$  and  $Y$ . Finally, define the computably composite structures  $\mathcal{M} := \mathcal{H}[\mathbf{M}]$  and  $\mathcal{N} := \mathcal{H}[\mathbf{N}]$ .

For every  $z \in H$ ,  $\mathcal{M}_z$  and  $\mathcal{N}_z$  are isomorphic, so the computably composite structures  $\mathcal{M}$  and  $\mathcal{N}$  are isomorphic.

**Proposition 31.** The isomorphisms from  $\mathcal{M}$  to  $\mathcal{N}$  are exactly the maps  $h_X \cup \bigcup_{z \in H} \psi_z$ , where  $X \in [\omega]^{<\omega}$  and  $\psi_z : \mathcal{M}_z \cong \mathcal{N}_{h_X(z)}$  for each  $z \in H$ .

*Proof.* This follows directly from Proposition 11 and Corollary 25. ■

The rest of this section discusses the Turing degrees of these isomorphisms and the isomorphism spectrum  $\text{IsoSpec}(\mathcal{M}, \mathcal{N})$ .

**Lemma 32.** Let  $\vartheta : A_0 \cong B_0$ . Then there is an isomorphism  $\varrho : \mathcal{M} \cong \mathcal{N}$  such that  $\varrho \equiv_T \vartheta$ .

*Proof.* Since  $\mathcal{M}_\emptyset = \{\emptyset\} \times A_0$  and  $\mathcal{N}_\emptyset = \{\emptyset\} \times B_0$ , the function  $\widehat{\vartheta}(\emptyset, a) := (\emptyset, \vartheta(a))$  is an isomorphism from  $\mathcal{M}_\emptyset$  to  $\mathcal{N}_\emptyset$  and  $\widehat{\vartheta} \equiv_T \vartheta$ . Define the isomorphism  $\varrho = h_X \cup \bigcup_{z \in H} \psi_z$  as follows. Take  $X = \emptyset$ , so  $h_X = h_\emptyset = id_H$ . We need to define each  $\psi_z : \mathcal{M}_z \cong \mathcal{N}_{h_X(z)} = \mathcal{N}_z$ :

- (1) For  $z = (i, a) \in \omega \times \{0, 1\}$ ,  $\mathcal{M}_{(i,a)} = \mathcal{N}_{(i,a)}$ , so we may take  $\psi_{(i,a)} := id_{\mathcal{M}_{(i,a)}}$ .
- (2) For  $z = Y \in [\omega]^{<\omega}$  with  $|Y|$  even we have  $\alpha_{Y,\emptyset} : \mathcal{M}_Y \cong \mathcal{M}_\emptyset$  and  $\beta_{\emptyset,Y} : \mathcal{N}_\emptyset \cong \mathcal{N}_Y$ , so we may define  $\psi_Y := \beta_{\emptyset,Y} \circ \widehat{\vartheta} \circ \alpha_{Y,\emptyset}$ , then  $\psi_Y : \mathcal{M}_Y \cong \mathcal{N}_Y = \mathcal{N}_{h_X(Y)}$ .
- (3) For  $z = Y \in [\omega]^{<\omega}$  with  $|Y|$  odd we have  $\beta_{Y,\emptyset} : \mathcal{M}_Y \cong \mathcal{N}_\emptyset$  and  $\alpha_{\emptyset,Y} : \mathcal{M}_\emptyset \cong \mathcal{N}_Y$ , so we may define  $\psi_Y := \alpha_{\emptyset,Y} \circ \widehat{\vartheta}^{-1} \circ \beta_{Y,\emptyset}$ , then  $\psi_Y : \mathcal{M}_Y \cong \mathcal{N}_Y = \mathcal{N}_{h_X(Y)}$ .

Since  $\alpha_{X,Y}$  and  $\beta_{X,Y}$  (as in Definition 30) are uniformly computable in  $X$  and  $Y$ , the collection  $\{\psi_Y : Y \in [\omega]^{<\omega}\}$  is uniformly computable in  $\widehat{\vartheta}$  and  $\psi_Y \equiv_T \widehat{\vartheta}$  for each  $Y$ . Thus,  $\widehat{\vartheta} \equiv_T \bigcup_{z \in H} \psi_z$ . Since  $h_\emptyset = id_H$  is computable, we have  $\varrho = id_\emptyset \cup \bigcup_{z \in H} \psi_z \equiv_T \widehat{\vartheta} \equiv_T \vartheta$ . ■

**Lemma 33.** Fix  $i < \omega$  and let  $\vartheta: \mathcal{A}_{i+1} \cong \mathcal{B}_{i+1}$ . Then there is an isomorphism  $\varrho: \mathcal{M} \cong \mathcal{N}$  such that  $\varrho \equiv_T \vartheta$ .

*Proof.* Since  $\mathcal{M}_{i,0} = \{(i, 0)\} \times \mathcal{A}_{i+1}$  and  $\mathcal{M}_{i,1} = \{(i, 1)\} \times \mathcal{B}_{i+1}$ , the function  $\widehat{\vartheta}((i, 0), a) := ((i, 1), \vartheta(a))$  is an isomorphism from  $\mathcal{M}_{(i,0)}$  to  $\mathcal{M}_{(i,1)}$  and  $\widehat{\vartheta} \equiv_T \vartheta$ . Define the isomorphism  $\varrho = h_X \cup \bigcup_{z \in H} \psi_z$  as follows. Take  $X = \{i\}$ , so  $h_X := h_{\{i\}}$ , which permutes  $(i, 0)$  with  $(i, 1)$ , permutes each  $Y \in [\omega]^{<\omega}$  with  $Y \triangle \{i\}$ , and fixes all  $(j, a)$  where  $j \neq i$ . We need to define each  $\psi_z: \mathcal{M}_z \cong \mathcal{N}_{h_{\{i\}}(z)}$ .

- (1) We have  $\mathcal{N}_{h_{\{i\}}(i,0)} = \mathcal{N}_{(i,1)} = \mathcal{M}_{(i,1)}$  and  $\mathcal{N}_{h_{\{i\}}(i,1)} = \mathcal{N}_{(i,0)} = \mathcal{M}_{(i,0)}$ , so we may take  $\psi_{(i,0)} := \widehat{\vartheta}$  and  $\psi_{(i,1)} := \widehat{\vartheta}^{-1}$ .
- (2) For  $j \neq i$  and  $a \in \{0, 1\}$ ,  $\mathcal{N}_{h_{\{i\}}(j,a)} = \mathcal{N}_{(j,a)} = \mathcal{M}_{(j,a)}$ , so let  $\psi_{(j,a)} := id_{\mathcal{M}_{(j,a)}}$ .
- (3) For  $|Y|$  even, take  $\psi_Y := \alpha_{Y, Y \triangle \{i\}}: \mathcal{M}_Y \cong \mathcal{N}_{Y \triangle \{i\}} = \mathcal{N}_{h_{\{i\}}(Y)}$ .
- (4) For  $|Y|$  odd, take  $\psi_Y := \beta_{Y, Y \triangle \{i\}}: \mathcal{M}_Y \cong \mathcal{N}_{Y \triangle \{i\}} = \mathcal{N}_{h_{\{i\}}(Y)}$ .

Then all  $\psi_z$  are uniformly computable in  $z$ , except for  $\psi_{(i,0)} = \widehat{\vartheta}$  and  $\psi_{(i,1)} = \widehat{\vartheta}^{-1}$ . Since  $h_i$  is computable,  $\varrho = h_{\{i\}} \cup \bigcup_{z \in H} \psi_z \equiv_T \widehat{\vartheta} \equiv_T \vartheta$ . ■

**Lemma 34.** Let  $\varrho: \mathcal{M} \cong \mathcal{N}$  be any isomorphism. Then there is an  $n < \omega$  and  $\vartheta: \mathcal{A}_n \cong \mathcal{B}_n$  such that  $\varrho \geq_T \vartheta$ .

*Proof.* By Proposition 31,  $\varrho = h_X \cup \bigcup_{z \in H} \psi_z$  for some  $X \in [\omega]^{<\omega}$  and  $\psi_z: \mathcal{M}_z \cong \mathcal{N}_{h_X(z)}$  for each  $z \in H$ . If  $X = \emptyset$ , then  $h_X = id_H$ , so  $\varrho \geq_T \varrho \upharpoonright_{M_\emptyset} = \psi_\emptyset: \mathcal{M}_\emptyset \cong \mathcal{N}_\emptyset$ . Since  $\mathcal{M}_\emptyset$  and  $\mathcal{N}_\emptyset$  are computably isomorphic to  $\mathcal{A}_0$  and  $\mathcal{B}_0$  respectively,  $\varrho$  computes an  $\mathcal{A}_0 \cong \mathcal{B}_0$  isomorphism. Now suppose  $X \neq \emptyset$  and pick any  $i \in X$ . Then  $h_X$  permutes  $(i, 0)$  and  $(i, 1)$ , so  $\varrho \geq_T \varrho \upharpoonright_{M_{(i,0)}} = \psi_{(i,0)}: \mathcal{M}_{(i,0)} \cong \mathcal{N}_{h_X(i,0)} = \mathcal{N}_{(i,1)}$ . Since  $\mathcal{M}_{(i,0)}$  and  $\mathcal{N}_{(i,1)}$  are computably isomorphic to  $\mathcal{A}_{i+1}$  and  $\mathcal{B}_{i+1}$  respectively,  $\varrho$  computes an  $\mathcal{A}_{i+1} \cong \mathcal{B}_{i+1}$  isomorphism. ■

We can now describe the isomorphism spectrum of  $\mathcal{M}$  and  $\mathcal{N}$  in terms of the isomorphism spectra of their components:

**Proposition 35.**  $\text{IsoSpec}(\mathcal{M}, \mathcal{N}) = \bigcup_{n < \omega} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i)$ .

*Proof.* It follows from Lemmas 32 and 33 that  $\bigcup_{i < \omega} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i) \subseteq \text{IsoSpec}(\mathcal{M}, \mathcal{N})$ . It follows from Lemma 34 that  $\text{IsoSpec}(\mathcal{M}, \mathcal{N}) \subseteq \bigcup_{n < \omega} \text{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i)$ . ■

As  $\{\mathcal{A}_i : i < \omega\}$  and  $\{\mathcal{B}_i : i < \omega\}$  were taken to be arbitrary, Theorem 29 follows. □

### 3.3 An isomorphism spectrum that is not finitely generated

Recall that we say that an isomorphism spectrum is *finitely generated* if it is equal to the upward-closure of a finite set of Turing degrees. We apply Theorem 29 to results of Thomason [12] to construct an isomorphism spectrum that is not finitely generated.

We will consider the computable copies  $(\omega, <_X)$  of  $(\omega, <)$  from Example 1 that encode c.e. sets  $X \subseteq \omega$ . Recall that for each c.e. set  $X$ , there is a unique isomorphism  $f: (\omega, <) \cong (\omega, <_X)$  and  $f \equiv_T X$ . We use a result of Thomason [12] to construct a uniformly computable family of copies of  $(\omega, <)$  so that the collection of isomorphisms between these structures and the usual copy,  $(\omega, <)$ , have pairwise disjoint Turing degrees.

**Theorem 36** (Thomason, [12]). There is a sublattice of the c.e. degrees isomorphic to the lattice of all finite sets of natural numbers.

To prove this, Thomason constructs a uniformly c.e. collection of sets  $\{X_n : n < \omega\}$ , each with a fixed computable enumeration. For each  $n < \omega$  and finite  $F \subseteq \omega$ , they define  $X_F := \bigoplus_{m \in F} X_m$ ,  $\mathbf{d}_n := \deg_T(X_n)$ , and  $\mathbf{d}_F := \deg_T(X_F) = \bigoplus_{m \in F} \mathbf{d}_m$  and show that for all  $F, G \in [\omega]^{<\omega}$ ,

- (1)  $\mathbf{d}_F \leq \mathbf{d}_G \Leftrightarrow F \subseteq G$ ,
- (2)  $\mathbf{d}_F \oplus \mathbf{d}_G = \mathbf{d}_{F \cup G}$ ,
- (3)  $\mathbf{d}_F \wedge \mathbf{d}_G$  exists and  $\mathbf{d}_F \wedge \mathbf{d}_G = \mathbf{d}_{F \cap G}$ .

**Remark 37.** The collection  $\{(\omega, <_{X_n}) : n < \omega\}$  is uniformly computable, and the set of degrees  $\{\mathbf{d}_n : n < \omega\}$  is pairwise incomparable. Moreover  $\mathbf{d}_n = \deg_T(X_n)$  is the Turing degree of the unique isomorphism from  $(\omega, <)$  to  $(\omega, <_{X_n})$ .

**Theorem 38.** Define the uniformly computable collections  $\mathbf{A} := \{\mathcal{A}_n : n < \omega\}$  and  $\mathbf{B} := \{\mathcal{B}_n : n < \omega\}$  where for all  $n < \omega$ ,  $\mathcal{A}_n = (\omega, <)$ , and  $\mathcal{B}_n = (\omega, <_{X_n})$ . Let  $\mathcal{M}$  and  $\mathcal{N}$  be the two computable copies from Theorem 29. Then  $\text{IsoSpec}(\mathcal{M}, \mathcal{N})$  is not finitely generated.

*Proof.* By Theorem 29,  $\text{IsoSpec}(\mathcal{M}, \mathcal{N}) = \bigcup_{n < \omega} \text{IsoSpec}((\omega, <), (\omega, <_{X_n})) = \bigcup_{n < \omega} \mathcal{D}_{\geq}(\mathbf{d}_n)$ , which is not finitely generated since the degrees  $\{\mathbf{d}_n : n < \omega\}$  are pairwise incomparable.  $\square$

## 4 Automorphism spectra of computably composite structures

Harizanov, Miller & Morozov [8] have considered similar questions about *automorphism spectra*, which have some interesting connections to our results. We would like to thank an anonymous referee for bringing this work to our attention.

**Definition 39** (Harizanov, Miller & Morozov, [8, Definition 1.1]). Let  $\mathcal{M}$  be a computable structure. The *automorphism spectrum* of  $\mathcal{M}$  is the set  $\text{AutSp}^*(\mathcal{M}) = \{\deg_T(f) : f \in \text{Aut}(\mathcal{M}) \setminus \{1_{\mathcal{M}}\}\}$  where  $1_{\mathcal{M}}$  is the identity automorphism on  $\mathcal{M}$ .

They also consider the upward-closures of automorphism spectra under  $\leq_T$  for which we will write  $\text{AutSp}_{\geq}^*(\mathcal{M}) = \{\mathbf{d} : \exists f \in \text{Aut}(\mathcal{M}) \setminus \{1_{\mathcal{M}}\} [f \leq_T \mathbf{d}]\}$ . Thus,  $\text{AutSp}_{\geq}^*(\mathcal{M})$  is equal to  $\text{IsoSpec}(\mathcal{M}, \mathcal{M})$  with the omission of the degrees that only compute the trivial isomorphism  $1_{\mathcal{M}} : \mathcal{M} \cong \mathcal{M}$ . It is known that the upward-closure of an automorphism spectrum is itself an automorphism spectrum:

**Proposition 40** (Harizanov, Miller & Morozov, [8, Proposition 7.1]). Let  $\mathcal{A}$  be any computable nonrigid structure. Then there exists a computable structure  $\mathcal{C}$  such that  $\text{AutSp}^*(\mathcal{C}) = \text{AutSp}_{\geq}^*(\mathcal{A})$ .

No automorphism spectrum has the form  $\{\mathbf{d}_1, \mathbf{d}_2\}$  where  $\mathbf{d}_1$  and  $\mathbf{d}_2$  are incomparable (see [8, Proposition 6.1]). Since there are incomparable degrees  $\mathbf{d}_1$  and  $\mathbf{d}_2$  such that  $\{\mathbf{d}_1\}$  and  $\{\mathbf{d}_2\}$  are automorphism spectra (see [8, Proposition 2.1]), the class of automorphism spectra is not closed under even finite unions. However, the following observation uses a uniform infinitary version of the proof of [8, Lemma 4.1] to show that the class of *upward-closures* of automorphism spectra is closed under computable union, satisfying the analog of our Theorem 29:

**Observation 41.** If  $\mathbf{A} = \{\mathcal{A}_i : i < \omega\}$  is a uniformly computable collection of copies and  $\mathcal{O}$  is the structure with domain  $\omega$  and unary relations  $U_n^{\mathcal{O}} = \{n\}$  for each  $n < \omega$ , then  $\text{AutSp}_{\geq}^*(\mathcal{O}[\mathbf{A}]) = \bigcup_{i < \omega} \text{AutSp}_{\geq}^*(\mathcal{A}_i)$ .

## 5 Categoricity spectra of some computably composite structures

We conclude by considering some basic properties of the categoricity spectra of computably composite structures on  $\mathcal{H}$ . We will consider structures of the form  $\mathcal{H}[\{z\} \times \mathcal{A} : z \in H]$ , which we will denote by  $\mathcal{H}[\mathcal{A}]$ , and their computable copies up to computable isomorphism. That is, we are concerned with the case where all components are isomorphic to a particular computable structure  $\mathcal{A}$ . For example, the structures in Theorem 38 are computable copies of  $\mathcal{H}[(\omega, <)]$ .

We may assume that computable copies of  $\mathcal{H}[\mathcal{A}]$  have the form  $\mathcal{H}[\mathbf{B}]$  where  $\mathbf{B} = \{\mathcal{B}_z : z \in H\}$  is a uniformly computable collection of copies of  $\mathcal{A}$ . The isomorphisms from  $\mathcal{H}[\mathcal{A}]$  to  $\mathcal{H}[\mathbf{B}]$  are the functions  $\varrho = h_X \cup \bigcup_{z \in H} \psi_z$  where  $X \in [\omega]^{<\omega}$  and  $\psi_z : \{z\} \times \mathcal{A} \cong \mathcal{B}_{h_X(z)}$  for all  $z \in H$ . The structure  $\mathcal{H}[\mathcal{A}]$  is  $\mathbf{d}$ -computably categorical if and only if  $\mathbf{d}$  computes some such  $\varrho$  for every uniformly computable collection of copies  $\{\mathcal{B}_z : z \in H\}$ . That is,  $\mathcal{A}$  must be  $\mathbf{d}$ -computably categorical in a uniform way.

**Definition 42.** We will say that  $\mathcal{A}$  is *uniformly  $\mathbf{d}$ -computably categorical* if there is a partial  $\mathbf{d}$ -computable binary function  $g$  such that whenever  $\varphi_e$  is the atomic diagram of a copy  $\hat{\mathcal{A}}$  of  $\mathcal{A}$ ,  $g(e, \cdot)$  is defined and is an isomorphism from  $\mathcal{A}$  to  $\hat{\mathcal{A}}$ . We define the *uniform categoricity spectrum* of  $\mathcal{A}$  and *degree of uniform categoricity* of  $\mathcal{A}$  in the analogous ways.

It is not hard to see that if  $\mathcal{A}$  is uniformly  $\mathbf{d}$ -computably categorical, then  $\mathcal{H}[\mathcal{A}]$  is  $\mathbf{d}$ -computably categorical. In some examples, the opposite implication holds as well:

**Example 43.** The simplest case is when  $\mathcal{A}$  is uniformly computably categorical, so  $\mathcal{H}[\mathcal{A}]$  is computably categorical. Then the categoricity spectrum of  $\mathcal{H}[\mathcal{A}]$  and the uniform categoricity spectrum of  $\mathcal{A}$  are both  $\mathcal{D}$ , the set of all Turing degrees.

**Example 44.** The structure  $(\omega, <)$  has strong degree of categoricity  $\mathbf{0}'$  and also uniform degree of categoricity  $\mathbf{0}'$ . Every isomorphism between the copies  $\mathcal{H}[(\omega, <)]$  and  $\mathcal{H}[(\omega, <_{\emptyset'})]$  as defined in Example 1 computes  $\mathbf{0}'$ , so  $\mathcal{H}[(\omega, <)]$  has strong degree of categoricity  $\mathbf{0}'$ , as do the structures in Theorem 38.

By the same argument, if  $\mathcal{A}$  has strong degree of categoricity  $\mathbf{d}$  and uniform degree of categoricity  $\mathbf{d}$ , then  $\mathcal{H}[\mathcal{A}]$  has strong degree of categoricity  $\mathbf{d}$ .

**Example 45.** The structure  $(\omega, S)$  where  $S(n) = n + 1$  is computably categorical, but has uniform degree of categoricity  $\mathbf{0}'$ . We can construct a uniformly computable collection of copies  $\{\mathcal{A}_n : n < \omega\}$  each with universe  $\omega$ , such that the  $S$ -least element of  $\mathcal{A}_n$  is 0 if and only if  $n \notin \emptyset'$ . Then defining  $\alpha: H \rightarrow \omega$  so that  $\alpha(X) = 0$  for  $X \in [\omega]^{<\omega}$  and  $\alpha(i, a) = i + 1$  for  $i < \omega$  and  $a < 2$ , the collection  $\{\{z\} \times \mathcal{B}_{\alpha(z)} : z \in H\}$  is uniformly computable. So  $\mathcal{H}[\{z\} \times \mathcal{B}_{\alpha(z)} : z \in H]$  is a computable copy of  $\mathcal{H}[\mathcal{A}]$ , but every isomorphism between these two copies computes  $\emptyset'$ . So  $\mathcal{H}[(\omega, S)]$  also has strong degree of categoricity  $\mathbf{0}'$ .

However, it appears that the full power of uniform  $\mathbf{d}$ -computable categoricity of  $\mathcal{A}$  may not be required for  $\mathcal{H}[\mathcal{A}]$  to be  $\mathbf{d}$ -computably categorical, since the following weakening is sufficient:

**Definition 46.** If  $\mathbf{C} = \{\mathcal{C}_i : i < \omega\}$  is a uniformly computable collection of copies of  $\mathcal{A}$ , we say that  $\mathcal{A}$  is *uniformly  $\mathbf{d}$ -computably categorical within  $\mathbf{C}$*  if there is a total  $\mathbf{d}$ -computable binary function  $h$  such that for all  $i < \omega$ ,  $h(i, \cdot)$  is an isomorphism from  $\mathcal{A}$  to  $\mathcal{C}_i$ .

**Proposition 47.** The structure  $\mathcal{H}[\mathcal{A}]$  is  $\mathbf{d}$ -computably categorical if and only if  $\mathcal{A}$  is uniformly  $\mathbf{d}$ -computably categorical within  $\mathbf{C}$  for every uniformly computable collection  $\mathbf{C}$  of copies of  $\mathcal{A}$ .

*Proof.* Suppose  $\mathcal{H}[\mathcal{A}]$  is  $\mathbf{d}$ -computably categorical and let  $\mathbf{C} = \{\mathcal{C}_i : i < \omega\}$  be a uniformly computable collection of copies of  $\mathcal{A}$ . Define  $\alpha: H \rightarrow \omega$  so that  $\alpha(X) = 0$  for all  $X \in [\omega]^{<\omega}$ , and  $\alpha(i, a) = i + 1$  for  $i < \omega$  and  $a < 2$ . Then  $\{\{z\} \times \mathcal{C}_{\alpha(z)} : z \in H\}$  is a pairwise disjoint uniformly computable collection of copies of  $\mathcal{A}$ . Since  $\mathcal{H}[\mathcal{A}]$  is  $\mathbf{d}$ -computably categorical, there is a  $\mathbf{d}$ -computable isomorphism  $\varrho: \mathcal{H}[\mathcal{A}] \cong \mathcal{H}[\{z\} \times \mathcal{C}_{\alpha(z)} : z \in H]$  and there is an  $X \in [\omega]^{<\omega}$  such that  $\varrho \upharpoonright_{\{\emptyset\} \times \mathcal{A}}: \{\emptyset\} \times \mathcal{A} \cong \{X\} \times \mathcal{C}_0$  and  $\varrho \upharpoonright_{\{(i,0)\} \times \mathcal{A}}: \{(i,0)\} \times \mathcal{A} \cong \{(i, X(i))\} \times \mathcal{C}_{i+1}$  for each  $i < \omega$ . Now define a function  $h$  so that  $h(0) := \pi_2(\varrho \upharpoonright_{\{\emptyset\} \times \mathcal{A}}(X))$  and  $h(i+1) := \pi_2(\varrho \upharpoonright_{\{(i,0)\} \times \mathcal{A}}(i, X(i)))$ , where  $\pi_2$  is the projection onto the second coordinate. Then  $h$  is total  $\mathbf{d}$ -computable and  $h(i)$  is an isomorphism from  $\mathcal{A}$  to  $\mathcal{C}_i$  for all  $i < \omega$ .

Now suppose that  $\mathcal{A}$  is uniformly  $\mathbf{d}$ -computably categorical within  $\mathbf{C}$  for every uniformly computable collection  $\mathbf{C}$  of copies of  $\mathcal{A}$  and let  $\mathcal{H}[\mathcal{B}_z : z \in H]$  be an arbitrary computable copy of  $\mathcal{H}[\mathcal{A}]$ . Fix a computable bijection  $\eta: \omega \rightarrow H$ . Then  $\{\mathcal{B}_{\eta(i)} : i < \omega\}$  is uniformly computable, so there is a total  $\mathbf{d}$ -computable function  $h$  such that  $h(i)$  is an isomorphism from  $\mathcal{A}$  to  $\mathcal{B}_{\eta(i)}$ . Define a function  $\varrho: \mathcal{H}[\mathcal{A}] \rightarrow \mathcal{H}[\mathcal{B}_z : z \in H]$  so that for  $z \in H$  and  $a \in A$ ,  $\varrho(z) := z$  and  $\varrho(z, a) := h(\eta^{-1}(z), a)$ . Then  $\varrho \upharpoonright_{\{z\} \times \mathcal{A}}$  is an isomorphism from  $\{z\} \times \mathcal{A}$  to  $\mathcal{B}_{\eta(\eta^{-1}(z))} = \mathcal{B}_z$ . Thus,  $\varrho$  is a  $\mathbf{d}$ -computable isomorphism from  $\mathcal{H}[\mathcal{A}]$  to  $\mathcal{H}[\mathcal{B}_z : z \in H]$ .  $\square$

In the previous examples,  $\mathcal{A}$  was uniformly  $\mathbf{d}$ -computably categorical exactly when  $\mathcal{A}$  was uniformly  $\mathbf{d}$ -computably categorical within every uniformly computable collection of copies. It is not clear whether this is true in general:

**Question 48.** Is there a computable structure  $\mathcal{A}$  and a Turing degree  $\mathbf{d}$  such that  $\mathcal{A}$  is not uniformly  $\mathbf{d}$ -computably categorical, but is uniformly  $\mathbf{d}$ -computably categorical within every uniformly computable collection of copies of  $\mathcal{A}$ ?

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