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Extending orders to types

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Abstract. Given an ordered structure, we study a natural way to extend the order to preorders on type spaces. For definably complete, linearly ordered structures, we give a characterisation of the preorder on the space of 1-types. We apply these results to the divisibility preorder on the space of ultrafilters on the set of natural numbers, giving an independence result about the suborder consisting of ultrafilters with only one fixed prime divisor, as well as a classification of ultrafilters with finitely many prime divisors.

1 Introduction

Let $(M, \leq, \ldots)$ be an expansion of an infinite partial order. In this note we study a construction extending $\leq$ to preorders on type spaces, mainly focusing on the case where M is a definably complete linear order.

More in detail, for A a set of parameters in a partially ordered structure $(M, \leq, \ldots)$, and k a natural number, we equip $S_k(A)$ with a preorder, denoted by $\precsim$, that is defined as follows. For two k-types p and q, we write $p \precsim q$ iff, for some realisations $\alpha \models p$ and $\beta \models q$ in an elementary extension $\mathfrak{U}$, we have $\alpha \leq \beta$ in the induced product (partial) order on $\mathfrak{U}^k$.

Equipping type spaces with relations is not a new idea. In particular, the definition we will use is equivalent to one that has already been considered in the past for arbitrary relations, see for instance [9, Definition 1.3] and references in that article. Another natural preorder, that we will not study here, may be defined by using tensor products, see [11]. In a different direction, a certain relational structure on type spaces is key to Hrushovski's *definability patterns* [6].

The properties of the relation $\precsim$, at the general level of partially ordered structures, are studied in Section 2. In Section 3 we study the posets $S_k(A)/{\approx}$, where $\approx$ is the equivalence relation induced by $\precsim$, in the case of linearly ordered, definably complete structures (Definition 17). Definably complete dense linear orders were introduced in [8]; under these assumptions, analogues of many theorems from real analysis hold [3; 4]. Examples of (not necessarily dense) definably complete linear orders include every expansion of $(\mathbb{N}, \leq)$, or of $(\mathbb{R}, \leq)$, and every o-minimal structure.

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Our main result is the following characterisation.

Main Theorem 1 (Theorem 26)**.** Let $(M, \leq, \ldots)$ be a definably complete expansion of a linear order and, for $A \subseteq M$, let $\mathrm{CC}(A)$ be the set of cuts in the definable closure of A that are filled in some elementary extension, with its natural linear order. Then $(S_1(A)/\approx) \cong \mathrm{CC}(A)$.

The original motivation for this work comes from the study of divisibility and congruences on the space $\beta\mathbb{N}$ of ultrafilters on the set of natural numbers, see e.g., [12; 2]. When the starting partial order is divisibility of natural numbers, and $\mathbb{N}$ is viewed as a structure with a symbol for every subset of each $\mathbb{N}^k$, the relations $\precsim$, $\approx$ induced on $S_1(\mathbb{N}) \cong \beta\mathbb{N}$ are denoted by $\tilde{|}$ and $=_\sim$ respectively. If p is prime (Definition 30), we write $\mathcal{E}_p$ for the restriction of the poset $(\beta\mathbb{N}/=_\sim, \tilde{|})$ to equivalence classes of ultrafilters of the form $\mathrm{tp}(\gamma^\delta/\mathbb{N})$, for $\gamma \models p$ and δ some nonstandard integer. In Section 4 we see that the study of $\mathcal{E}_p$ naturally corresponds to studying $\precsim$ on $S_1(\gamma)$ (Remark 32), and deduce the following.

Main Corollary 2 (Theorems 33 and 37)**.** For every prime ultrafilter p, if $\gamma \models p$ then the order $\mathcal{E}_p$ is isomorphic to $\mathrm{CC}(\gamma)$. It is independent of ZFC whether all $\mathcal{E}_p$ with p nonprincipal are isomorphic.

If q is the type of an increasing k-tuple of primes, one can similarly look at the suborder $\mathcal{E}_q$, whose study reduces to the study of $\precsim$ on $S_k(\gamma)$. Although, for $k \geq 2$, a structure theorem for $\mathcal{E}_q$ is still lacking, we provide a classification of its points.

We conclude by leaving some open questions in Section 5.

2 Types in ordered structures

Let $(M, \leq, \ldots)$ be an infinite structure expanding a partial order. We also denote by $\leq$ the induced product order on its Cartesian powers M^k; namely, $(a_1, \ldots, a_k) \leq (b_1, \ldots, b_k)$ iff for every $i \leq k$ we have $a_i \leq b_i$.

We adopt standard model-theoretic conventions. We fix a sufficiently large cardinal κ, and work inside a κ-saturated and κ-strongly homogeneous monster model $\mathfrak{U}$. *Small* means "of cardinality less than κ". Parameter sets, usually denoted by A, are assumed to be small subsets of $\mathfrak{U}$, unless otherwise indicated. The relation of having the same type over A is denoted by $\equiv_A$. Realisations of types over small sets are assumed to be in $\mathfrak{U}$. We sometimes write x for a tuple of variables $x = (x_1, \ldots, x_k)$, and similarly for tuples of elements. *Definable* means "definable with parameters"; similarly, formulas are allowed to use parameters from sets that will be clear from context. By *type* we will mean "complete type over some parameter set" but, as usual, *type-definable* means "definable by a partial type". If $p \in S_k(A)$ and f is an A-definable function, by f_*p we denote the pushforward of p along f. We write dcl for the definable closure operator.

2.1 Ordering types

Definition 3. Let $A \subseteq M$ and $k \in \mathbb{N}$. For $p, q \in S_k(A)$, we define $p \lessapprox q$ iff there are $\alpha \models p$ and $\beta \models q$ such that $\alpha \leq \beta$. We write $p \approx q$ iff $p \lessapprox q \lessapprox p$.

Remark 4. By an automorphism argument, $p \lessapprox q$ if and only if for every $\alpha \models p$ there is $\beta \models q$ such that $\alpha \leq \beta$, if and only if for every $\beta \models q$ there is $\alpha \models p$ such that $\alpha \leq \beta$.

Remark 5. Clearly, $S_k(A)/\approx$ is partially ordered by the relation induced by $\precsim$, which we also denote by the same symbol. Whenever we mention $S_k(A)/\approx$, we consider it equipped with this partial order.

We call a formula $\varphi(x_1, \ldots, x_k)$ *upward closed* (respectively, *downward closed*) iff the set $\varphi(M) \subseteq M^k$ it defines is upward closed (respectively, downward closed).

Proposition 6. For every $p, q \in S_k(A)$, the following are equivalent.

(1) $p \lessapprox q$.
(2) If $\varphi(x) \in p(x)$ is upward closed, then $\varphi(x) \in q(x)$.
(3) If $\varphi(x) \in q(x)$ is downward closed, then $\varphi(x) \in p(x)$.

All conditions above also imply the following, and are equivalent to it provided that $k = 1$ or that $\mathrm{dcl}(A)$ contains two comparable points.

(4) For all A-definable, increasing partial functions $f\colon M^k \to M$ such that both p and q contain the domain of f, we have $f_*p \lessapprox f_*q$.

Proof. (1) ⇒ (2)**.** Immediate from the definitions.

(2) ⇒ (1)**.** Suppose that $p \not\lessapprox q$. Then, by compactness, there are $\vartheta(x) \in p(x)$ and $\psi(y) \in q(y)$ such that $\vartheta(x) \wedge \psi(y) \vdash x \not\leq y$. Let $\varphi(y)$ be the upward closure of $\vartheta(y)$, namely, $\varphi(y) \coloneqq \exists z\,(\vartheta(z) \wedge z \leq y)$. Then, $\varphi(y)$ is inconsistent with $\psi(y)$, hence cannot belong to $q(y)$. But $\vartheta(x)$ implies $\varphi(x)$, hence the latter belongs to $p(x)$.

(2) ⇔ (3)**.** This follows from the observation that the complements of upward closed sets are precisely the downward closed sets.

(1) ⇒ (4)**.** Immediate from the definitions.

(4) ⇒ (1) in the case $k = 1$. Apply (4) to the identity function.

(4) ⇒ (2) if $\mathrm{dcl}(A)$ contains two comparable points. Fix two elements $0 < 1 \in \mathrm{dcl}(A)$, and use them to write characteristic functions of definable sets, with value 1 on points belonging to the set. For upward closed sets such characteristic functions are increasing. □

We will say that p *lies on an antichain* if some definable set in p is an antichain.

Proposition 7. Let $p \in S_k(A)$. The following are equivalent.

(1) There are no $\alpha \neq \alpha' \models p$ such that $\alpha \leq \alpha'$.
(2) p lies on an antichain.

Proof. If (1) holds, by compactness and up to taking conjunctions there is $\varphi(x) \in p(x)$ such that $\varphi(x) \wedge \varphi(y) \vdash x = y \vee x \not\leq y$. Hence $\varphi(x)$ defines an antichain, proving (1) ⇒ (2), and (2) ⇒ (1) is obvious. □

Proposition 8. For every $p \in S_k(A)$, the following are equivalent.

(1) The $\approx$-class of p is a singleton.
(2) The locus of p is convex, that is, if $\alpha, \alpha' \models p$ and $\alpha \leq \beta \leq \alpha'$ then $\beta \models p$.

Proof. For (1) ⇒ (2), if $\alpha, \alpha' \models p$ and β is such that $\alpha \leq \beta \leq \alpha'$, then $\mathrm{tp}(\beta/A) \approx p$ by definition, hence by (1) we must have $\beta \models p$. For (2) ⇒ (1), assume that $p \approx q$. By Remark 4 there are $\alpha, \alpha' \models p$ and $\beta \models q$ such that $\alpha \leq \beta \leq \alpha'$. It then follows from (2) that $p = q$. □

Proposition 9. If p lies on an antichain, then its $\approx$-class is a singleton.

Proof. Proposition 7 (1) clearly implies Proposition 8 (2). □

It can happen that the $\approx$-class of a type p is a singleton even if p does not lie on an antichain.

Example 10. In $(\mathbb{Q}, <)$, or more generally in an arbitrary o-minimal structure, every 1-type lies in a singleton $\approx$-class, but linearity of the order implies that there are no antichains with more than one point.

Remark 11. Later we will deal with a structure where, for every type p, and every pair $\alpha < \alpha'$ of realisations of p, there is $\beta \not\models p$ with $\alpha \leq \beta \leq \alpha'$. By Propositions 7 to 9, this condition implies that the $\approx$-class of p is a singleton if and only if p lies on an antichain.

It may happen that some interval $[\alpha, \alpha']$ consists entirely of realisations of a type p, but the locus of p is not convex. This is visible in the following example.

Example 12. Consider the structure $M = (\mathbb{Q}, \leq, X)$, where $X = \bigcup_{m\in\mathbb{Z}}\{a \in \mathbb{Q} : 2m + \pi < a < 2m + 1 + \pi\}$. An easy back-and-forth argument shows that the theory of this structure eliminates quantifiers after naming, for each $n \in \omega$, the relation $R_n(x, y)$ that holds iff the set $\{k \in \mathbb{Z} : x < k + \pi < y\}$ has size n. Note that $R_0(x, y) \wedge R_0(y, x)$ is an equivalence relation, and each equivalence class is either a maximal convex subset of X or a maximal convex subset of $\mathbb{Q}\backslash X$. Moreover, if $x < y$, then $R_n(x, y)$ holds precisely when $[x, y]$ meets $n + 1$ equivalence classes.

It follows from quantifier elimination that there are precisely two elements of $S_1(M)$ that are larger than M, one in X, call it p, and one in its complement, call it q. If $\alpha, \alpha' \models p$ and $\models R_0(\alpha, \alpha')$, then $[\alpha, \alpha']$ consists entirely of realisations of p. Yet, the locus of p is not convex, since if $\alpha, \alpha' \models p$ and $\models R_2(\alpha, \alpha')$ then there is $\beta \models q$ with $\alpha < \beta < \alpha'$.

Proposition 13. For $p \in S_k(A)$, the following are equivalent.

(1) There are $\alpha, \alpha' \models p$ such that, for every $i \leq k$, we have $\alpha_i < \alpha'_i$.
(2) Let $\alpha \models p$. For $i \leq k$, there are infinite intervals $I_i = (\gamma_i, \delta_i)$, definable over $\mathfrak{U}$, containing α_i, with both (γ_i, α_i) and (α_i, δ_i) infinite, and such that, for all $\beta \in I_1 \times \ldots \times I_k$, we have $\mathrm{tp}(\beta/A) \approx p$.

Proof. (1) $\Rightarrow$ (2)**.** By an automorphism argument, given an arbitrary $\alpha \models p$, we can find α' as in (1). By this and compactness, we can then find a sequence $(\alpha^j : j \in \mathbb{Z} \cup \{-\infty, +\infty\})$ of realisations of p with $\alpha^0 = \alpha$ and such that if $j < \ell$ then $\alpha^j_i < \alpha^\ell_i$. It now suffices to set $I_i \coloneqq (\alpha_i^{-\infty}, \alpha_i^{+\infty})$.

(2) $\Rightarrow$ (1)**.** Fix $\alpha \models p$ and let the I_i be given by the assumption. Let $\beta \in I_1 \times \ldots \times I_k$ be such that, for every $i \leq k$, we have $\alpha_i < \beta_i$. As $\mathrm{tp}(\beta/A) \approx p$, there is $\alpha' \models p$ such that $\beta \leq \alpha'$, and for all $i \leq k$ we have $\alpha_i < \beta_i \leq \alpha'_i$. □

2.2 Linear orders

Remark 14. If $(M, \leq)$ is a linear order, then

(a) $S_1(A)/{\approx}$ is linear, and
(b) if $\mathrm{dcl}(A) \neq \varnothing$, then $S_k(A)/{\approx}$ is linear if and only if $k = 1$. In fact, if $k \geq 2$, $a \in \mathrm{dcl}(A)$, $b \leq a \leq c$, and $b \neq c$, then the types of $(b, b, \ldots, b, c)$ and $(c, c, \ldots, c, b)$ over A are incomparable.

Proposition 15. Let $(M, <)$ be a linear order, $D \subseteq M^k$ and, for $i \in \{1, \ldots, k\}$,

$$G_i \coloneqq \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, x_i) : (x_1, x_2, \ldots, x_k) \in D\}.$$

The following conditions are equivalent.

(1) D is an antichain.
(2) For every $i \in \{1, 2, \ldots, k\}$, the set G_i is the graph of a partial strictly decreasing function.
(3) For some $i \in \{1, 2, \ldots, k\}$, the set G_i is the graph of a partial strictly decreasing function.

Proof. For (1) $\Rightarrow$ (2), fix a coordinate, without loss of generality x_k. If $(a_1, \ldots, a_k) \in D$ and $a \neq a_k$, by linearity $a < a_k$ or $a > a_k$, hence $(a_1, \ldots, a_{k-1}, a) \notin D$ and D can be regarded as the graph of a partial function of $x_1, \ldots, x_{k-1}$; call it f. Let us check that f is strictly decreasing. If $(a_1, \ldots, a_{k-1}) < (b_1, \ldots, b_{k-1})$, as D is an antichain, we must have $(a_1, \ldots, a_{k-1}, f(a_1, \ldots, a_{k-1})) \not\leq (b_1, \ldots, b_{k-1}, f(b_1, \ldots, b_{k-1}))$, and since the order on M is linear it follows that $f(a_1, \ldots, a_{k-1}) > f(b_1, \ldots, b_{k-1})$. The implication (2) $\Rightarrow$ (3) is obvious, and the proof of (3) $\Rightarrow$ (1) is easy. □

Corollary 16. If p is a k-type in a linear order, the following conditions are equivalent.

(1) p lies on an antichain.
(2) For some/every $i \in \{1, 2, \ldots, k\}$, the set

$$\{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, x_i) : (x_1, x_2, \ldots, x_k) \models p\}$$

is the graph of a partial strictly decreasing function.

Proof. (1) $\Rightarrow$ (2)**.** If p lies on an antichain, its realisations form an antichain and we apply Proposition 15.

(2) $\Rightarrow$ (1)**.** By Proposition 15 the set of realisations of p is an antichain, and we apply compactness. □

3 Definably complete structures

Definition 17. A linearly ordered structure M is *definably complete* iff, for every nonempty definable subset X of M, if X has an upper bound then it has a supremum, and if X has a lower bound then it has an infimum.

Remark 18.

(a) Whether M is definably complete only depends on its theory; that is, if $M \equiv N$, then M is definably complete if and only if N is.
(b) If $X \subseteq M$ is A-definable and $\inf(X)$ exists, then $\inf(X) \in \mathrm{dcl}(A)$.

Assumption 19. For the rest of the section, unless otherwise stated, we assume $(M, \leq, \ldots)$ to be a linearly ordered, definably complete structure.

Definition 20.

(a) A *cut* in a linear order $(X, \leq)$ is a pair (L, R) of subsets of X such that $L \cup R = X$ and for all $\ell \in L$ and $r \in R$ we have $\ell \leq r$.
(b) If $(X, \leq) \subseteq (Y, \leq)$ are linear orders and (L, R) is a cut in X, we say that $b \in Y$ *realises* (L, R) iff $L < b < R$ or $\{b\} = L \cap R$.

Observe that one of L, R may be empty, and that L and R are either disjoint or intersect in a unique point. Cuts of the second kind are precisely those that are realised by exactly one point in every linear order in which X embeds. Clearly, the set of realisations of a given cut is always a convex set.

Definition 21. For $p \in S_1(A)$, we define

$$L_p \coloneqq \{a \in \mathrm{dcl}(A) : p(x) \vdash x \geq a\},$$
$$R_p \coloneqq \{a \in \mathrm{dcl}(A) : p(x) \vdash x \leq a\}.$$

The *cut* of p is the type-definable set $\{x = a\}$ if $L_p \cap R_p = \{a\}$, or the type-definable set $\{x > a : a \in L_p\} \cup \{x < a : a \in R_p\}$ if $L_p \cap R_p = \varnothing$.

Lemma 22. Let $p(x) \in S_1(A)$, and let $C \subseteq \mathfrak{U}$ be the set of realisations of its cut. Then, p is realised coinitially and cofinally many times in C.

Proof. If p is realised in $\mathrm{dcl}(A)$ then C is a singleton and the conclusion is trivial, hence we may assume it is not. In particular, because in linearly ordered structures algebraic closure coincides with definable closure, p must have infinitely many realisations.

Observe that every realisation of p in $\mathfrak{U}$ must be in C. It suffices to show that, for every $\beta \in C$, there are realisations of p to its left and to its right. We prove the first statement; the second one follows by considering the structure with the reverse order.

It is enough to prove that $p(x) \cup \{x < \beta\}$ is consistent. By compactness, it suffices to show that if $\varphi(x)$ is a formula in $p(x)$, then $\varphi(x) \wedge (x < \beta)$ is consistent. Since $\varphi(x)$ belongs to a type, it defines a nonempty set.

If $L_p \neq \varnothing$, up to taking a conjunction with a formula in $p(x)$ bounded from below, by definable completeness we may assume that $\varphi(\mathfrak{U})$ has an infimum a, which belongs to $\mathrm{dcl}(A)$ by Remark 18 (b). Because $\varphi(x) \in p(x)$, we must have $a \in L_p$, for otherwise p would be inconsistent. In particular, $a < \beta$, hence in $\mathfrak{U}$, by definition of infimum, there is a point of $\varphi(\mathfrak{U})$ in the interval $[a, \beta)$.

If instead $L_p = \varnothing$, then we are assuming that p is the type of a point smaller than $\mathrm{dcl}(A)$ and what we have to show is that $\varphi(\mathfrak{U})$ has arbitrarily small points. If not, by definable completeness it has an infimum in $\mathrm{dcl}(A)$. But then $\varphi(x)$ cannot be consistent with being smaller than $\mathrm{dcl}(A)$. □

Example 23. Assuming definable completeness is necessary. For instance, if M is the expansion of $(\mathbb{Q}, \leq)$ by a predicate P for $\{x \in \mathbb{Q} : x < \pi\}$, then the type $p \in S_1(M)$ defined as $p(x) = \{P(x)\} \cup \{a < x : a \in P(\mathbb{Q})\}$ is not realised cofinally in its cut.

Definition 24. The space $\mathrm{CC}(A)$ of *consistent cuts* over A is the space of quantifier-free 1-types in the language $\{\leq\}$ over $\mathrm{dcl}(A)$ (where dcl is computed in the original structure, not in the reduct to $\{\leq\}$). We view it as a linear order in the natural way.

Remark 25. If A is a small set, $\mathrm{CC}(A)$ can be identified with the set of cuts in $\mathrm{dcl}(A)$ which are realised by some element of $\mathfrak{U}$. Equivalently, this is the space of ultrafilters on the Boolean algebra of subsets of $\mathfrak{U}$ generated by intervals with endpoints in $\mathrm{dcl}(A)$.

Theorem 26. The natural map $S_1(A)/{\approx} \to \mathrm{CC}(A)$, sending an $\approx$-class to its cut, is an isomorphism of linear orders.

Proof. If the cut of p lies below the cut of q, by definition there is $a \in \mathrm{dcl}(A)$ such that $p(x) \vdash x < a$ and $q(x) \vdash x \geq a$, or such that $p(x) \vdash x \leq a$ and $q(x) \vdash x > a$. Hence, every realisation of p lies below every realisation of q, witnessing that $p \not\gtrsim q$, and in particular $p \not\approx q$. It follows that the map $\pi \colon S_1(A)/{\approx} \to \mathrm{CC}(A)$ sending the class of a 1-type p over A to the cut of p in $\mathrm{dcl}(A)$ is well-defined and increasing. Moreover, it is injective by Lemma 22, and surjective by Remark 25. □

Example 27. The analogue of Theorem 26 in higher dimension, with $S_k(A)$ in place of $S_1(A)$ and with $\mathrm{CC}(A)$ replaced by the space of quantifier-free k-types in the language $\{\leq\}$ over $\mathrm{dcl}(A)$, is false. For instance, in the full theory on $\mathbb{N}$ (see the beginning of Section 4), let γ be an infinite point and work over the prime model $\mathbb{N}(\gamma)$ over γ. Let α be a realisation of the cut just after $\mathbb{N}$, that is, the cut $(\mathbb{N}, \mathrm{dcl}(\{\gamma\})\backslash\mathbb{N})$. Then $\mathrm{tp}((\alpha, \gamma - \alpha)/\mathbb{N}(\gamma))$ and $\mathrm{tp}((\alpha, \gamma - \alpha - 1)/\mathbb{N}(\gamma))$ have the same quantifier-free part in the language $\{\leq\}$, but $(\alpha, \gamma - \alpha)$ lies on the antichain $\{(x, y) : x + y = \gamma\}$, hence the $\approx$-class of its type is a singleton by Proposition 9.

By Proposition 6, if $\mathrm{dcl}(A)$ contains two comparable points, then one can characterise $\approx$-classes by using increasing A-definable partial functions $\mathfrak{U}^k \to \mathfrak{U}$. We leave it open (Problem 43) to give a more explicit description of the poset $S_k(A)/{\approx}$.

4 Divisibility by finitely many primes

We now come to the original motivation of this work, namely, divisibility between ultrafilters. View $\mathbb{N}$ as a structure in the richest possible language,[1] namely the one containing, for every k, a symbol for every relation or function on $\mathbb{N}^k$. Observe that ultrafilters in $\beta\mathbb{N}^k$ are naturally identified with k-types over $\mathbb{N}$.

Write ${}^*\mathbb{N}$ (instead of $\mathfrak{U}$) for a monster model of this theory. By Remark 18 (a) and the fact that $\mathbb{N}$ is (obviously) definably complete with respect to its natural order, so is ${}^*\mathbb{N}$. As the language at hand trivially has

[1] Up to having the same definable sets.

built-in Skolem functions, the definable closure of every set $A \subseteq {}^*\mathbb{N}$ is a model, the *prime model* $\mathbb{N}(A)$ over A. For γ a single element, write $\mathbb{N}(\gamma)$ instead of $\mathbb{N}(\{\gamma\})$. Every point of $\mathbb{N}(\gamma)$ is of the form $f(\gamma)$ for a suitable $f\colon \mathbb{N} \to \mathbb{N}$.

Remark 28. Up to isomorphism, $\mathbb{N}(\gamma)$ is the ultrapower of $\mathbb{N}$ along the ultrafilter corresponding to $\mathrm{tp}(\gamma/\mathbb{N})$. In nonstandard-analytical parlance, we say that γ is a *generator* of this ultrafilter.

Remark 29. Distinct tuples with the same type over an arbitrary parameter set must have infinite distance, since if γ and δ differ on the coordinate i and $|\gamma_i - \delta_i| = n$ then the remainder class of γ_i modulo $n+1$ differs from that of δ_i.

Henceforth, the symbols $\precapprox$ and $\approx$ will be used for the relations induced by the usual order $\leq$ on $\mathbb{N}$. For the corresponding relations induced by divisibility we will, consistently with previous papers on the subject, use the notations $\mathrel{\tilde{|}}$ and $=_\sim$ respectively.

Definition 30. An ultrafilter is *prime* iff it contains the set $\mathbb{P}$ of primes. The set of such ultrafilters is denoted by $\overline{\mathbb{P}}$.

Definition 31. Let $q \in S_k(\mathbb{N})$ be the type of an increasing k-tuple of primes, that is,

$$q(x_1, \ldots, x_k) \vdash (x_1 < \ldots < x_k) \wedge \textstyle\bigwedge_{i \leq k}(x_i \in \mathbb{P}).$$

We denote by $\mathcal{E}_q$ the poset of $=_\sim$-equivalence classes of ultrafilters of the form $\mathrm{tp}(\gamma_1^{\alpha_1} \cdot \ldots \cdot \gamma_k^{\alpha_k}/\mathbb{N})$, for $\gamma \models q$ and $\alpha_1, \ldots, \alpha_k$ some nonstandard integers.

We emphasise that some γ_i may generate the same prime ultrafilter, and that being twice divisible by p is not the same as being divisible by p^2; for example, if $p = \mathrm{tp}(\gamma_1/\mathbb{N}) = \mathrm{tp}(\gamma_2/\mathbb{N})$ but $\gamma_1 \neq \gamma_2$, then $\mathrm{tp}(\gamma_1 \cdot \gamma_2/\mathbb{N}) \neq \mathrm{tp}(\gamma_1^2/\mathbb{N})$, the former is twice divisible by p, and the latter is divisible by p^2.

4.1 Prime powers

Šobot [14, Question 6.1] asked if the orders $\mathcal{E}_p$ are isomorphic for all $p \in \overline{\mathbb{P}} \backslash \mathbb{P}$. We show in Theorem 37 that the answer is independent of ZFC, by applying to $M = \mathbb{N}(\gamma)$ the results obtained in the previous sections.

Remark 32. In order to study $\mathcal{E}_p$, suppose $\alpha, \beta \in {}^*\mathbb{N}$ have a unique prime factor, and this unique prime factor has type p. Since p is prime, if α', β' witness that $\mathrm{tp}(\alpha/\mathbb{N}) \mathrel{\tilde{|}} \mathrm{tp}(\beta/\mathbb{N})$, then there must be $\gamma \models p$ and $\eta_0 \leq \eta_1$ such that $\alpha' = \gamma^{\eta_0}, \beta' = \gamma^{\eta_1}$. If furthermore $\mathrm{tp}(\beta/\mathbb{N}) \mathrel{\tilde{|}} \mathrm{tp}(\alpha/\mathbb{N})$ then there must be some $\eta_2 \equiv_\gamma \eta_0$ (so, $\gamma^{\eta_2} \models \mathrm{tp}(\alpha/\mathbb{N})$) such that $\eta_1 \leq \eta_2$. In other words, the map sending the $=_\sim$-class of $\mathrm{tp}(\gamma^\eta/\mathbb{N})$ to the $\approx$-class of $\mathrm{tp}(\eta/\mathbb{N}(\gamma))$ is an isomorphism between $\mathcal{E}_p$ and $S_1(\gamma)/\approx$.

Below, whenever we mention CC, it will always be in the sense of the order $\leq$ on ${}^*\mathbb{N}$.

Theorem 33. For every $p \in \overline{\mathbb{P}} \subseteq \beta\mathbb{N}$, if $\gamma \models p$, then $\mathcal{E}_p \cong \mathrm{CC}(\gamma)$.

Proof. By Remark 32 and Theorem 26. □

Example 34. If $p \in \mathbb{P}$ is a standard prime, then $\mathcal{E}_p \cong \omega + 1$.

It follows immediately from Theorem 33 that relations in the Rudin–Keisler preorder yield embeddings between different $\mathcal{E}_p$.[2]

[2] Recall that an ultrafilter p is Rudin–Keisler-below an ultrafilter p' iff there is a function f such that $p = f_* p'$.

Corollary 35.

(1) If p is Rudin–Keisler-below p', then there is an embedding of $\mathcal{E}_p$ into $\mathcal{E}_{p'}$.
(2) If p and p' are Rudin–Keisler-equivalent, then $\mathcal{E}_p \cong \mathcal{E}_{p'}$.

Proof. (1). This follows from the fact that if p is Rudin–Keisler below p' then p is realised in the prime model over any realisation of p'.

(2). Let $f\colon \mathbb{N} \to \mathbb{N}$ be a bijection such that $f_*p = p'$, which exists by [5, Theorem 8.17]. For every $\gamma \models p$, if $\delta = f(\gamma)$ then $\delta \models p'$. We have

$$\begin{aligned}\{g(\delta) : g\colon \mathbb{N} \to \mathbb{N}\} &= \{g(f(\gamma)) : g\colon \mathbb{N} \to \mathbb{N}\} \\ &= \{h(\gamma) : h\colon \mathbb{N} \to \mathbb{N}\},\end{aligned}$$

or in other words $\mathbb{N}(\gamma) = \mathbb{N}(\delta)$, and the conclusion follows. □

Lemma 36. If $A \subseteq {}^*\mathbb{N}$, then an element of $\mathrm{CC}(A)$ is realised in $\mathrm{dcl}(A)$ if and only if it has an immediate successor in $\mathrm{CC}(A)$.

Proof. Let (L, R) be a cut. Because $\mathrm{dcl}(A) = \mathbb{N}(A)$ is a model, its order type is that of $\mathbb{N}$ followed by copies of $\mathbb{Z}$. In particular, in $\mathbb{N}(A)$, the only point with no immediate predecessor is 0, and moreover it is not consistent to add points between a and $a+1$. From this, it is easy to see that (L, R) is realised in $\mathbb{N}(A)$ if and only if R has a minimum a, if and only if (L, R) has an immediate successor, namely the cut realised by $a+1$. □

As usual, $\mathfrak{c}$ denotes the cardinality of the continuum. For $p \in \beta\mathbb{P}$, we denote by $\mathbb{N}^{\mathbb{P}}/p$ the ultrapower of $\mathbb{N}$ along p.

Theorem 37.

(1) If the continuum hypothesis holds, then all the $\mathcal{E}_p$ with p nonprincipal are isomorphic.
(2) $\mathsf{ZFC} + \neg\mathsf{CH}$ does not prove that all $\mathcal{E}_p$ with p nonprincipal are isomorphic. More precisely, let M be a model of $\mathsf{ZFC} + \neg\mathsf{CH}$. In every forcing extension of M obtained by adding $\kappa \geq \mathfrak{c}^M$-many Cohen reals, there are $p, q \in \overline{\mathbb{P}}\backslash\mathbb{P}$ such that $\mathcal{E}_p \ncong \mathcal{E}_q$.

Proof. (1). By Theorem 33 it suffices to show that all $\mathrm{CC}(\gamma)$ with $\gamma \notin \mathbb{N}$ are isomorphic. The reduct of $\mathbb{N}(\gamma)$ to $\{\leq\}$ is a discrete order with a minimum and no maximum, and eliminates quantifiers after naming 0 and the successor and predecessor functions. From this it is easy to see that $\mathrm{CC}(\gamma) \cong S_1(\mathbb{N}(\gamma) \restriction \{\leq\})$. Now, the ultrapower of a countable discrete order with a minimum and no maximum over every nonprincipal ultrafilter on $\mathbb{N}$ is $\aleph_1$-saturated of size $\leq \mathfrak{c}$. By standard model theoretic facts about saturated models, under CH there is a unique such model up to isomorphism.

(2). By [10, Theorem, p. 95] in such a forcing extension, for every regular uncountable $\lambda \leq \kappa$, there is an ultrafilter p_λ on $\mathbb{N}$ such that $\mathbb{N}^{\mathbb{N}}/p_\lambda$ has cofinality λ.[3] By fixing a bijection between $\mathbb{N}$ and $\mathbb{P}$ we may assume that p_λ is an ultrafilter on $\mathbb{P}$. We show that $\mathcal{E}_{p_{\aleph_1}} \ncong \mathcal{E}_{p_{\aleph_2}}$. Assume not. Let I_i be the suborder of $\mathcal{E}_{p_{\aleph_i}}$ consisting of points that have an immediate successor. As we are assuming $\mathcal{E}_{p_{\aleph_1}} \cong \mathcal{E}_{p_{\aleph_2}}$, we have $I_1 \cong I_2$. Let $\gamma_i \models p_{\aleph_i}$. By Theorem 33, $\mathcal{E}_{p_{\aleph_i}} \cong \mathrm{CC}(\gamma_i)$, and by Lemma 36 the subset of the latter consisting of points with an immediate successor is isomorphic to $\mathrm{dcl}(\gamma_i) = \mathbb{N}(\gamma_i)$, that is, by Remark 28, to $\mathbb{N}^{\mathbb{P}}/p_{\aleph_i}$. Hence, from $I_1 \cong I_2$ we obtain $\mathbb{N}^{\mathbb{P}}/p_{\aleph_1} \cong \mathbb{N}^{\mathbb{P}}/p_{\aleph_2}$, contradicting that they have different cofinalities. □

[3] Independently, a similar result was proved by Canjar, see [1, Section 3.4].

4.2 Finitely many prime divisors

Let us now look at the case $k \geq 2$.

Remark 38. Let q be the type of an increasing k-tuple of primes, and suppose that the classes of p, p' are in $\mathcal{E}_q$. If p, p' are comparable, there must be $\gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_k, \varepsilon_1, \ldots, \varepsilon_k$ such that

$$(\gamma_1, \ldots, \gamma_k) \models q, \qquad \alpha \coloneqq \gamma_1^{\delta_1} \cdot \ldots \cdot \gamma_k^{\delta_k} \models p, \qquad \beta \coloneqq \gamma_1^{\varepsilon_1} \cdot \ldots \cdot \gamma_k^{\varepsilon_k} \models p',$$

and $\alpha \mid \beta$. It follows that, in order for p, p' to be comparable, it is necessary for their squarefree parts to be equal, analogously to what we say in Remark 32. By coding $\{\gamma_1, \ldots, \gamma_k\}$ with $\gamma \coloneqq \gamma_1 \cdot \ldots \cdot \gamma_k$ (or by fixing a bijection between $\mathbb{N}$ and $\mathbb{P}^{[k]}$), we therefore see that $\mathcal{E}_q$ is isomorphic to $(S_k(\gamma), \precsim)/\approx$, hence we reduce to studying the latter.

Remark 39. Analogous considerations show that if q and q' are types of increasing finite tuples of primes, of possibly different length, then we have two mutually exclusive cases:

(a) q is the pushforward of q' along the projection on a subset of the coordinates, or vice versa; or
(b) every point of $\mathcal{E}_q$ is incomparable to every point of $\mathcal{E}_{q'}$.

We now give a classification of the points of $\mathcal{E}_q$.

Remark 40. If $(\alpha_1, \ldots, \alpha_k) < (\alpha'_1, \ldots, \alpha'_k)$ are realisations of $p \in S_k(\gamma)$ and i is such that $\alpha_i < \alpha'_i$ then, by Remark 29, the point $(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_k)$ does not realise p and lies between them. Therefore, by Remark 11 and Proposition 9 the $\approx$-class of p is a singleton if and only if p lies on an antichain.

Fix a k-type q of an increasing sequence of primes. Let p be an ultrafilter whose $=_\sim$-class is in $\mathcal{E}_q$. Fix $\alpha \models p$ of the form $\alpha = \prod_{i \leq k} \gamma_i^{\delta_i}$, where $(\gamma_1, \ldots, \gamma_k) \models q$, and let $\gamma \coloneqq \gamma_1 \cdot \ldots \cdot \gamma_k$. Then, we have five mutually exclusive cases.

(a) All δ_i lie in $\mathbb{N}(\gamma)$, so there are functions $f_i \colon \mathbb{N} \to \mathbb{N}$ such that $\alpha = \prod_{i \leq k} \gamma_i^{f_i(\gamma)}$. Clearly, such a $=_\sim$-class must be a singleton.
(b) No δ_i lies in $\mathbb{N}(\gamma)$ and, for every $i \leq k$, there is a strictly decreasing, definable partial function $g_i \colon \mathbb{N}(\gamma)^{k-1} \to \mathbb{N}(\gamma)$ such that $\delta_i = g_i(\delta_1, \ldots, \delta_{i-1}, \delta_{i+1}, \ldots, \delta_k)$. By Corollary 16 and Remark 40, the class of p is a singleton.
(c) There is a product $I_1 \times \ldots \times I_k$ of infinite intervals containing $(\delta_1, \ldots, \delta_k)$ such that all $I_i \cap \{x < \delta_i\}$ and all $I_i \cap \{x > \delta_i\}$ are infinite and such that, for every $(\varepsilon_1, \ldots, \varepsilon_k) \in I_1 \times \ldots \times I_k$, the type of $\prod_{i \leq k} \gamma_i^{\varepsilon_i}$ is $=_\sim$-equivalent to p.
(d) No δ_i lies in $\mathbb{N}(\gamma)$, the class of p is not a singleton, and we are not in case (c). Then, by Proposition 13, whenever $\delta' \equiv_\gamma \delta$ is such that $\delta \geq \delta'$, there is $i \leq k$ such that $\delta_i = \delta'_i$.
(e) For some $\ell > 0$ there are indices, say $i = \ell + 1, \ell + 2, \ldots, k$ (up to a permutation of the variables), such that $\delta_i \in \mathbb{N}(\gamma)$, and for $j \leq \ell$ we have $\delta_j \notin \mathbb{N}(\gamma)$. Then the class of p naturally corresponds to some class in $\mathcal{E}_{q'}$ of kind (b), (c), or (d), where q' is the projection of q on the first ℓ coordinates.

Example 41. Examples of each of these cases may be obtained as follows.

(a) Consider, e.g., $\gamma_1^{\gamma_1} \cdot \ldots \cdot \gamma_k^{\gamma_k}$.
(b) Take some pairwise distinct $\delta_1, \ldots, \delta_{k-1}$ not in $\mathbb{N}(\gamma)$ and all in Archimedean classes smaller than that of γ_1, and set $g_k(x_1, \ldots, x_{k-1}) = \gamma_1 - (x_1 + \ldots + x_{k-1})$.
(c) This happens for instance if the tuple $(\gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_k)$ is tensor with all coordinates infinite (see [7] for the definition of, and various results on, tensor tuples).

(d) Let $k = 2$, and take $\delta_1 > \mathbb{N}$ such that the pair (δ_1, γ) is tensor. Consider the interval $[\gamma \cdot (\gamma - \delta_1), \gamma \cdot (\gamma - \delta_1 + 1)]$. Take an arbitrary δ_2 in this interval but not in $\mathbb{N}(\{\delta_1, \gamma\})$. The type of $\gamma_1^{\delta_1}\gamma_2^{\delta_2}$ does not lie on an antichain, as we may take any $\delta_2' \equiv_{\delta_1,\gamma} \delta_2$ and consider $\gamma_1^{\delta_1}\gamma_2^{\delta_2'}$. On the other hand, whenever $(\delta_1'', \delta_2'') \equiv_\gamma (\delta_1, \delta_2)$ and $\delta_1'' > \delta_1$, by Remark 29 we also have $\delta_1'' > \delta_1 + 1$, and it follows that $\delta_2'' < \delta_2$.

(e) Set $\delta_{\ell+1} = \ldots = \delta_k = 1$ and choose $\delta_1, \ldots, \delta_\ell$ in such a way that $\gamma_1^{\delta_1} \cdot \ldots \cdot \gamma_\ell^{\delta_\ell}$ is of kind (b), (c), or (d).

Remark 42.

(a) It follows by inspection of the cases above that, if $\delta, \varepsilon \in {}^*\mathbb{N}^k$ are at finite distance, then the classes of $\mathrm{tp}(\prod_{i\leq k} \gamma_i^{\delta_i}/\mathbb{N})$ and of $\mathrm{tp}(\prod_{i\leq k} \gamma_i^{\varepsilon_i}/\mathbb{N})$ are of the same kind.

(b) Taking supremums may result in a change of kind. For example, let $u_n \coloneqq \mathrm{tp}(\gamma_1^1\gamma_2^n/\mathbb{N})$. For all n, the class of u_n is of kind (a), but the supremum of these classes is of kind (e).

5 Further directions

In Theorem 26 we gave a fairly explicit description of $S_1(A)/{\approx}$. The problem of finding a nice characterisation in the higher dimensional case still remains.

Problem 43. For A a subset of a linearly ordered, definably complete structure, describe the posets $S_k(A)/{\approx}$.

As for divisibility, even assuming a satisfactory answer to Problem 43, it remains open to give a precise description of how the various $\mathcal{E}_q$ fit together.

Furthermore, the case of ultrafilters divisible by infinitely many primes is still largely unexplored. Still, some results and techniques from this paper apply also to the case of infinitely many prime divisors; for instance, one can easily adapt the argument from Remark 40 and obtain that an ultrafilter lies on an antichain if and only if it has a singleton $=_\sim$-equivalence class.

A tool introduced to study this problem are so-called patterns, developed in [14; 13]. Patterns reflect the quantity of each element of $\mathcal{E}_p$ (for various $p \in \overline{\mathbb{P}}$) dividing the given ultrafilter. They do not determine completely the place of an ultrafilter in the divisibility preorder, nor are determined by it, but several divisibility-related properties have equivalent versions in terms of patterns.

Observe that the study of divisibility has an equivalent formulation.

Remark 44. There is an isomorphism between $(\mathbb{N}, |)$ and the poset of finite multisets of natural numbers with inclusion.

Problem 45. Describe the order on ultrafilters on finite multisets of natural numbers induced by inclusion.

A first natural subcase is that of finite sets, that is, the study of the relation $\precapprox$ induced by the partial order $\subseteq$ on $\mathcal{P}_{<\omega}(\mathbb{N})$, where the latter is equipped with the full language, namely with a symbol for every possible relation. This corresponds to divisibility amongst squarefree natural numbers. Even in this special case, one cannot argue anymore by fixing primes γ_i as in Section 4, as visible in the following examples.

Example 46. Let π be the increasing enumeration of the primes and let $\alpha > \mathbb{N}$. By Proposition 9, the type of $\prod_{i=\alpha}^{2\alpha} \pi(i)$ forms a non-realised singleton $=_\sim$-class, since it lies on the antichain $\left\{ \prod_{i=n}^{2n} \pi(i) : n \in \mathbb{N} \right\}$.

Example 47. With the same notation as in the previous example, all ultrafilters generated by $\prod_{i=1}^{\alpha} \pi(i)$, with $\alpha > \mathbb{N}$, are in the same $=_\sim$-class. In fact, for every $\alpha_1 < \alpha_2$, there is $\alpha_3 > \alpha_2$ with the same type as α_1; it follows that $\prod_{i=1}^{\alpha_1} \pi(i)$ and $\prod_{i=1}^{\alpha_3} \pi(i)$ have the same type, and clearly $\prod_{i=1}^{\alpha_1} \pi(i) \mid \prod_{i=1}^{\alpha_2} \pi(i) \mid \prod_{i=1}^{\alpha_3} \pi(i)$.

Let us conclude with a final example, phrased directly in terms of inclusion on hyperfinite sets. It shows that there are intervals with both endpoints not in $\mathbb{N}$ that do not form singleton classes.

Example 48. For fixed δ, all intervals of the form $(\delta-\gamma,\delta+\gamma)$ with (γ,δ) tensor are in the same $\approx$-class: to witness the inequality $(\delta-\varepsilon,\delta+\varepsilon)\gtrapprox(\delta-\gamma,\delta+\gamma)$ for $\varepsilon>\gamma$, just replace ε by some ε' such that the triple $(\varepsilon',\gamma,\delta)$ is tensor.

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