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Sunflowerable structures

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Abstract. We call an infinite structure $\mathcal{M}$ *sunflowerable* if whenever $\mathcal{M}'$ is isomorphic to $\mathcal{M}$ with underlying set M', consisting of finite sets of bounded size, there is an $M_0 \subseteq M'$ such that M_0 is a sunflower and $\mathcal{M}'{\restriction}_{M_0}$ is isomorphic to $\mathcal{M}$. We give sufficient conditions on $\mathcal{M}$ to show that $\mathcal{M}$ is sunflowerable. These conditions allow us to show that several well-known structures are sunflowerable and give a complete characterization of the countable linear orderings that are sunflowerable.

We show that a sunflowerable structure must be indivisible. This allows us to show that any Fraïssé limit that has the 3-disjoint amalgamation property and a single unary type must be indivisible.

In addition to studying sunflowerability of infinite structures we also consider an analogous property of an age, which we call the *sunflower property.* We show that any sunflowerable structure must have an age with the sunflower property. We also give concrete bounds in the case that the age has the hereditary property, the 3-disjoint amalgamation property and is indivisible.

1 Introduction

In combinatorial set theory a *sunflower*, or a Δ*-system*, is a collection of sets any two pairs of which have a common intersection. The existence of infinite sunflowers has important applications in mathematical logic, such as for forcing large generic structures (for more information see [8; 13; 3]).

There are two important situations when it is known that infinite sunflowers exist. First there is the following result of Erdős and Rado, which tells us that whenever we have an infinite collection of finite sets, all of the same size, that infinite collection must contain an infinite sunflower.

Theorem 1 (Erdős & Rado, [7])**.** Suppose $n \in \omega$ and Y an infinite set consisting of sets of size n. Then there is a sunflower $Y_0 \subseteq Y$ with $|Y_0| = |Y|$.

Proof. See [7, Theorem I.ii] or [12, p. 107 and p. 421]. □

Second we have the following, which is known as the Δ-system lemma, and which considers the case when all sets are finite, but not necessarily of the same size.

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Theorem 2 (Shanin)**.** Suppose Y is an uncountable collection of finite sets. Then there is a sunflower $Y_0 \subseteq Y$ with $|Y_0| = |Y|$.

Proof. See [11, Theorem I.9.18]. □

In this paper, we will be interested in collections of sets where the collection itself is an $\mathscr{L}$-structure, for some language $\mathscr{L}$, e.g. is a graph or a linear ordering. We say an infinite $\mathscr{L}$-structure $\mathcal{M}$ is *sunflowerable* if whenever there is an isomorphic copy of $\mathcal{M}$ whose underlying set consists of finite sets of the same size, then there is an isomorphic substructure whose underlying set forms a sunflower. In this set up, the Erdős and Rado theorem says that if $\mathscr{L}_\varnothing$ is the empty language then every infinite $\mathscr{L}_\varnothing$-structure is sunflowerable.

In Section 2 we will study variants of the property of being sunflowerable and will give a natural condition on an $\mathscr{L}$-structure, which is satisfied by dense linear orderings and the Rado graph, that guarantees a structure is sunflowerable.

In Section 3 we focus on linear orderings. For both κ-scattered linear orderings and countable linear orderings we give a complete characterization of when a structure is sunflowerable.

While the existence of infinite sunflowers is important in mathematical logic and combinatorial set theory, the existence of large finite sunflowers has significance in many other fields, including the study of circuit lower bounds, matrix multiplication, pseudo-randomness, and cryptography. For an overview of the connections to computer science, see [5]. One of the most important results in the study of finite sunflowers is the sunflower lemma of Erdős & Rado [7]. Ignoring the explicit bounds that were obtained, this lemma can be framed as follows.

Lemma 3. For every $n \in \omega$ and every finite set X there is a finite set Y such that for every set S and every injection $f\colon Y \to \mathcal{P}_n(S)$ there is a set $X' \subseteq Y$ where $f[X']$ is a sunflower and $|X'| = |X|$.

If this lemma holds when we replace "finite set" by "finite structure in an age $\mathbf{K}$" (in an appropriate way) then we say $\mathbf{K}$ has the sunflower property. In Section 4 we will consider the sunflower property of an age and show that if a countable structure is sunflowerable then its age has the sunflower property.

In Section 5 we also show that if a structure is sunflowerable, or if an age has the sunflower property, then it must be indivisible. This allows us to deduce that any structure that has a single unary type and the 3-disjoint amalgamation property is indivisible.

Finally in Section 6 we end with some conjectures.

1.1 Notation

For a set S, we let $\mathcal{P}_{<\omega}(S)$ be the collection of finite subsets of S. For $n \in \omega$, we let $\mathcal{P}_n(S)$ be the collection of subsets of S of size n. We also let $[n] = \{0, 1, \dots, n-1\}$. If f is a function, we let $\mathrm{dom}(f)$ denote its domain and $\mathrm{ran}(f)$ denote its range. If α is an ordinal, then α^* is the order type given by $(\alpha, \ni)$, i.e., the reverse ordering of $(\alpha, \in)$. Suppose α is an ordinal and for all $i \in \alpha$, $\boldsymbol{x}_i$ is a sequence. We let $\langle\!\langle \boldsymbol{x}_i \rangle\!\rangle_{i\in\alpha}$ denote the sequence that is the concatenation, in order, of each $\boldsymbol{x}_i$.

We fix a language $\mathscr{L}$ all of whose functions and relations have finite arity. We will use script letters to represent $\mathscr{L}$-structures, i.e., $\mathcal{A}$, $\mathcal{M}_0$, etc. and the corresponding Roman letters to represent their underlying sets, i.e., A, M_0, etc. If $\mathcal{A}$ is a structure and $B \subseteq A$ then we let $\mathcal{A}|_B$ be the substructure of $\mathcal{A}$ with underlying set B. We will use boldface to signify sequences of variables or of elements. We will write $\boldsymbol{a} \subseteq \mathcal{M}$ when $\boldsymbol{a}$ is a sequence of elements of $\mathcal{M}$. We will write $|\boldsymbol{a}|$ for the length of such a sequence. We write $\mathrm{qtp}_{\mathcal{M}}(\boldsymbol{a})$ for the *(complete) quantifier-free type* of $\boldsymbol{a}$ in $\mathcal{M}$, i.e., the collection of literals that are true of the tuple $\boldsymbol{a}$ in $\mathcal{M}$. All quantifier-free types considered in this paper will be complete. For a quantifier-free type p We will write $p(\boldsymbol{x})$ to indicate that the free variables of p are contained in $\boldsymbol{x}$. If $\boldsymbol{y}$ is contained in $\boldsymbol{x}$ then we write $p(\boldsymbol{x})|_{\boldsymbol{y}}$ to denote the collection of formulas in p that only use variables in $\boldsymbol{y}$.

Definition 4. We say a structure $\mathcal{M}$ is *ultrahomogeneous* if whenever

(1) $\boldsymbol{a}, \boldsymbol{b} \subseteq \mathcal{M}$ are tuples of size $< |M|$,
(2) $\mathrm{qtp}_{\mathcal{M}}(\boldsymbol{a}) = \mathrm{qtp}_{\mathcal{M}}(\boldsymbol{b})$ (and in particular $|\boldsymbol{a}| = |\boldsymbol{b}|$),
(3) $a \in \mathcal{M}$,

then there is some $b \in \mathcal{M}$ such that $\mathrm{qtp}_{\mathcal{M}}(\boldsymbol{a}, a) = \mathrm{qtp}_{\mathcal{M}}(\boldsymbol{b}, b)$.

2 Sunflowerable structures

In this section we introduce one of the main properties of study, that of being *sunflowerable*.

Definition 5. A *sunflower* is a set $Y \subseteq \mathcal{P}_{<\omega}(S)$ (for some set S) such that $(\exists r)\, (\forall p, q \in Y)\, p \neq q \rightarrow p \cap q = r$. We say a set $Z \subseteq \mathcal{P}_{<\omega}(S)$ (for some set S) *contains* a sunflower if there is a $Y \subseteq Z$ that is a sunflower.

Definition 6. Suppose $\mathcal{M}$ is an infinite $\mathscr{L}$-structure and $n \in \omega$. We say $\mathcal{M}$ is n*-sunflowerable* if, whenever S is a set and $f \colon M \rightarrow \mathcal{P}_n(S)$ is an injection, there is a subset $M' \subseteq M$ such that $\mathcal{M}{\restriction}_{M'} \cong \mathcal{M}$ and $f[M']$ is a sunflower. We say $\mathcal{M}$ is *sunflowerable* if it is n-sunflowerable for every $n \in \omega$. We say $\mathcal{M}$ is *strongly sunflowerable* if, whenever S is a set and $f \colon M \rightarrow \mathcal{P}_{<\omega}(S)$ is an injection, there is a subset $M' \subseteq M$ such that $\mathcal{M}{\restriction}_{M'} \cong \mathcal{M}$ and $f[M']$ is a sunflower.

Note that every structure is 0-sunflowerable and 1-sunflowerable. The following shows that the notion of being strongly sunflowerable is only interesting for uncountable structures.

Lemma 7. If $\mathcal{M}$ is countable, then $\mathcal{M}$ is not strongly sunflowerable.

Proof. Note that $\omega \subseteq \mathcal{P}_{<\omega}(\omega)$ but ω contains no sunflower of size 3. Therefore no structure on ω can contain an infinite substructure that is a sunflower. ☐

The following shows that the notions of n-sunflowerable form a hierarchy.

Lemma 8. Suppose $n \in \omega$ and $\mathcal{M}$ is $(n+1)$-sunflowerable. Then $\mathcal{M}$ is n-sunflowerable.

Proof. Suppose S is a set, $\mathcal{M}' \cong \mathcal{M}$ and $M' \subseteq \mathcal{P}_n(S)$. Let x be any set not in $\bigcup M'$, let $M^* = \{a \cup \{x\} : a \in M'\}$ and let $\alpha \colon M' \rightarrow M^*$ be the map where $\alpha(a) = a \cup \{x\}$. Then α is a bijection. Let $\mathcal{M}^*$ be the $\mathscr{L}$-structure such that α is an isomorphism from $\mathcal{M}'$ to $\mathcal{M}^*$. Because $\mathcal{M}$ is $(n+1)$-sunflowerable and $M^* \subseteq \mathcal{P}_{n+1}(S)$ there must be an $M^\circ \subseteq M^*$ such that M° is a sunflower and $\mathcal{M}^*{\restriction}_{M^\circ} \cong \mathcal{M}^*$. Let $M^+ = \alpha^{-1}[M^\circ]$. Then M^+ is a sunflower and $\mathcal{M}'{\restriction}_{M^+} \cong \mathcal{M}$. Therefore, as $\mathcal{M}'$ was arbitrary, $\mathcal{M}$ is n-sunflowerable. ☐

We now give some conditions on a structure, which will guarantee that it is sunflowerable.

Definition 9. We say a structure $\mathcal{M}$ is κ*-indivisible* if, whenever $(P_i)_{i\in\alpha}$ partitions M with $\alpha < \kappa$, for some $i \in \alpha$ there is a $Q \subseteq P_i$ such that $\mathcal{M}{\restriction}_Q \cong \mathcal{M}$. We say $\mathcal{M}$ is *indivisible* if it is $|M|$-indivisible.

Being indivisible is a strong form of robustness, which has a long history. For more on indivisibility (and closely related notions) see, for example, [4; 16; 10].

We first show that if a structure is 2-sunflowerable then it is indivisible.

Proposition 10. If $\mathcal{M}$ is 2-sunflowerable then $\mathcal{M}$ is indivisible.

Proof. Suppose $\mathcal{M}$ is 2-sunflowerable, $|M| = \kappa$ and $(P_i)_{i\in\alpha}$ is a partition of M with $\alpha < \kappa$. Let $|P_i| = \gamma_i$ for $i \in \alpha$. For $i \in \alpha$ let $Q_i = \{\{(0, i), (i+1, j)\}\}_{j\in\gamma_i} \subseteq \mathcal{P}_2(\kappa \times \kappa)$. Note $|Q_i| = |P_i|$ and Q_i is a sunflower. Let $Q = \bigcup_{i\in\alpha} Q_i$. Note that for $i < j \in \alpha$, $Q_i \cap Q_j = \varnothing$. Therefore if for $i \in \alpha$, $\beta_i \colon P_i \rightarrow Q_i$ is a bijection and

$\beta = \bigcup_{i\in\alpha} \beta_i$, then β is a bijection from the underlying set of $\mathcal{M}$ onto Q. Let $\mathcal{N}$ be the $\mathscr{L}$-structure such that $\beta\colon \mathcal{M} \to \mathcal{N}$ is an isomorphism.

Because $\mathcal{M}$ is 2-sunflowerable, there must be a $Y \subseteq Q$ that is a sunflower with $\mathcal{N}{\upharpoonright}_Y \cong \mathcal{M}$. Then $|Y| = |M| = \kappa$. But then there must be distinct $a, b \in Y$ with $a \cap b \neq \varnothing$, as otherwise $|Y \cap Q_i| = 1$ for all $i \in \alpha$ and $|Y| \leq \alpha < \kappa$. As $|a| = |b| = 2$, we must have $|a \cap b| = 1$. Hence, for some $i \in \alpha$ we have $a \cap b = \{(0, i)\}$. But, because Y is a sunflower, this implies for all distinct $c, d \in Y$ that $c \cap d = \{(0, i)\}$ and $c, d \in Q_i$. Therefore $Y \subseteq Q_i$. But then $\mathcal{M}{\upharpoonright}_{\beta^{-1}[Y]} \subseteq P_i$ is isomorphic to $\mathcal{M}$ and witnesses the fact that $\mathcal{M}$ is indivisible. $\square$

We now show that being strongly sunflowerable is equivalent to being sunflowerable and uncountable.

Proposition 11. The following are equivalent:

(1) $\mathcal{M}$ is sunflowerable and uncountable,
(2) $\mathcal{M}$ is strongly sunflowerable.

Proof. By Lemma 7 we have that (2) implies (1).

Now suppose (1). As $\mathcal{M}$ is sunflowerable, by Proposition 10 $\mathcal{M}$ is indivisible. Suppose S is a set, $\mathcal{M}' \cong \mathcal{M}$ and $M' \subseteq \mathcal{P}_{<\omega}(S)$. Let $c\colon M' \to \omega$ be such that $c(a) = |a|$. As $\mathcal{M}'$ is indivisible and uncountable, there must be an $M^* \subseteq M'$ such that $|c[M^*]| = 1$ and $\mathcal{M}'{\upharpoonright}_{M^*} \cong \mathcal{M}'$. But then, as $\mathcal{M}^*$ is sunflowerable there must be an $M^\circ \subseteq M^*$ such that $\mathcal{M}^*{\upharpoonright}_{M^\circ} \cong \mathcal{M}^*$ and M° is a sunflower. Therefore $\mathcal{M}$ is strongly sunflowerable and (2) holds. $\square$

Proposition 10 has consequences for the types of languages that admit 2-sunflowerable structures.

Lemma 12. Suppose $\mathcal{M}$ is an indivisible infinite $\mathscr{L}$-structure. Then $\mathscr{L}$ has no constants. In particular, if $\mathcal{M}$ is 2-sunflowerable, then $\mathscr{L}$ has no constants.

Proof. Suppose, to get a contradiction, that c is a constant in $\mathscr{L}$. Let $P_0 = \left\{c^{\mathcal{M}}\right\}$ and $P_1 = M \backslash P_0$. Then $\{P_0, P_1\}$ is a partition of M. As $\mathcal{M}$ is indivisible, there is an $i \in \{0, 1\}$ and an $\mathcal{N} \subseteq \mathcal{M}$ such that $\mathcal{N} \cong \mathcal{M}$ and N is contained in P_i. But, as $\mathcal{N} \subseteq \mathcal{M}$, we have $c^{\mathcal{N}} = c^{\mathcal{M}}$ and so, in particular, $c^{\mathcal{M}} \in N$. Therefore $N \not\subseteq P_1$. But $|P_0| = 1$ and so $N \not\subseteq P_0$, getting us our contradiction. The fact that $\mathscr{L}$ does not have constant symbols if $\mathcal{M}$ is sunflowerable then follows from Proposition 10. $\square$

We now give a property that will imply κ-indivisibility on κ-saturated structures.

Definition 13. Suppose $\mathcal{M}$ is an $\mathscr{L}$-structure, $\boldsymbol{a}$ is a tuple contained in $\mathcal{M}$, and a is an element of $\mathcal{M}$ not in $\boldsymbol{a}$. We then let $B_{\boldsymbol{a},a} = \{b \in \mathcal{M} : \mathrm{qtp}_{\mathcal{M}}(\boldsymbol{a}, a) = \mathrm{qtp}_{\mathcal{M}}(\boldsymbol{a}, b)\}$. We say $\mathcal{M}$ has *universal duplication of κ-quantifier-free types* if for all tuples $\boldsymbol{a} \subseteq \mathcal{M}$ of length $< \kappa$ and all $a \in \mathcal{M}\backslash\boldsymbol{a}$, there is a $C \subseteq B_{\boldsymbol{a},a}$ where $\mathcal{M}{\upharpoonright}_C \cong \mathcal{M}$. We say $\mathcal{M}$ has *strong universal duplication of κ-quantifier-free types* if for all sequences $\boldsymbol{a} \in \mathcal{M}$ of length $< \kappa$ and $a \in \mathcal{M}\backslash\boldsymbol{a}$, we have $\mathcal{M}{\upharpoonright}_{B_{\boldsymbol{a},a}} \cong \mathcal{M}$. If $\kappa = |M|$, then we omit mention of it.

Universal duplication of κ-quantifier-free types can been seen as a homogeneity requirement on the model. It is also worth noting that a structure having universal duplication of ω-quantifier-free types implies that all functions must be choice functions.

Lemma 14. Suppose $\mathcal{M}$ is an infinite $\mathscr{L}$-structure that has universal duplication of ω-quantifier-free types and f is a function symbol in $\mathscr{L}$ of arity n. Then $\mathcal{M} \models (\forall x_0, \ldots, x_{n-1}) \bigvee_{i\in[n]} f(x_0, \ldots, x_{n-1}) = x_i$.

Proof. Suppose, to get a contradiction, there is a tuple $\boldsymbol{a} = (a_0, \ldots, a_{n-1}) \subseteq \mathcal{M}$ such that $f(\boldsymbol{a}) \in \mathcal{M}\backslash\{\boldsymbol{a}\}$. Then, as $\mathcal{M}$ has universal duplication of ω-quantifier-free types, there is a $N \subseteq B_{\boldsymbol{a},f(\boldsymbol{a})}$ with $\mathcal{N} \cong \mathcal{M}$. But $B_{\boldsymbol{a},f(\boldsymbol{a})} = \{b : f(\boldsymbol{a}) = b\} = \{f(\boldsymbol{a})\}$. In particular, $\left|B_{\boldsymbol{a},f(\boldsymbol{a})}\right| = 1$ getting us our contradiction. $\square$

The following gives a basic condition when universal duplication of κ-quantifier-free types is the same as strong universal duplication of κ-quantifier-free types.

Definition 15. We say a structure $\mathcal{M}$ has 3-*amalgamation of quantifier-free types* if, whenever $p(\boldsymbol{x}, \boldsymbol{y})$, $q(\boldsymbol{y}, \boldsymbol{z}), r(\boldsymbol{z}, \boldsymbol{x})$ are quantifier-free types realized in $\mathcal{M}$ such that

$$\begin{aligned} p(\boldsymbol{x}, \boldsymbol{y})\restriction_{\boldsymbol{x}} &= r(\boldsymbol{z}, \boldsymbol{x})\restriction_{\boldsymbol{x}}, \\ p(\boldsymbol{x}, \boldsymbol{y})\restriction_{\boldsymbol{y}} &= q(\boldsymbol{y}, \boldsymbol{z})\restriction_{\boldsymbol{y}}, \\ q(\boldsymbol{y}, \boldsymbol{z})\restriction_{\boldsymbol{z}} &= r(\boldsymbol{z}, \boldsymbol{x})\restriction_{\boldsymbol{z}}, \end{aligned}$$

there is a quantifier-free type $a(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ realized in $\mathcal{M}$ such that

$$\begin{aligned} a(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\restriction_{\boldsymbol{xy}} &= p(\boldsymbol{x}, \boldsymbol{y}), \\ a(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\restriction_{\boldsymbol{yz}} &= q(\boldsymbol{y}, \boldsymbol{z}), \\ a(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\restriction_{\boldsymbol{zx}} &= p(\boldsymbol{z}, \boldsymbol{x}). \end{aligned}$$

The 3-amalgamation of quantifier-free types is a strengthening of the amalgamation property and, in general, there are notions of n-amalgamation for all n. These are closely related to the n-disjoint amalgamation property (n-DAP) of an age defined in Definition 40.

For our purposes, we will be focused on saturated structures that admit quantifier-elimination (and hence are ultrahomogeneous). In this situation 3-amalgamation implies strong universal duplication of quantifier-free types.

Lemma 16. Suppose $\mathcal{M}$ is $|M|$-saturated, has quantifier-elimination, has 3-amalgamation of quantifier-free types, and all elements of $\mathcal{M}$ have the same quantifier-free type. Then $\mathcal{M}$ has strong universal duplication of quantifier-free types.

Proof. Let $|M| = \kappa$ and let $(p_i(\boldsymbol{x}_i))_{i\in\kappa}$ be an enumeration of quantifier-free types realized in $\mathcal{M}$ over tuples of size $< \kappa$. For $i \in \kappa$, let $\eta_i = (\exists \boldsymbol{x}_i) \bigwedge p_i(\boldsymbol{x}_i)$ and let $I = \{(i,j) \in \kappa^2 : p_i(\boldsymbol{x}_i) \subseteq q_j(\boldsymbol{x}_j) \wedge \boldsymbol{x}_j = \boldsymbol{x}_i y_j\}$. For $(i,j) \in I$, let $\gamma_{i,j} = (\forall \boldsymbol{x}_i)\ [\bigwedge p_i(\boldsymbol{x}_i) \to (\exists y_j) \bigwedge q_j(\boldsymbol{x}_i y_j)]$. Note, as $\mathcal{M}$ is saturated with quantifier-elimination, $\mathcal{M} \models \bigwedge_{i\in\kappa} \eta_i$ and $\mathcal{M} \models \bigwedge_{(i,j)\in I} \gamma_{i,j}$. Further note that a standard back-and-forth argument shows that if $|\mathcal{N}| = |\mathcal{M}|$ and $\bigwedge_{i\in\kappa} \eta_i$ and $\bigwedge_{(i,j)\in I} \gamma_{i,j}$, then $\mathcal{N} \cong \mathcal{M}$.

Suppose, to get a contradiction, that $\mathcal{M}$ does not have strong universal duplication of quantifier-free types. Pick some $\boldsymbol{a}, a$ such that $\mathcal{M}\restriction_{B_{\boldsymbol{a},a}}$ is not isomorphic to $\mathcal{M}$. Let $r = \mathrm{qtp}_{\mathcal{M}}(\boldsymbol{a}, a)$. There must then be some $\boldsymbol{b} \in B_{\boldsymbol{a},a}$ and quantifier-free type $q(\boldsymbol{x}, y)$ such that $\mathcal{M} \models (\exists y)\, q(\boldsymbol{b}, y)$ and $\mathcal{M} \not\models (\exists y)\, r(\boldsymbol{a}, y) \wedge q(\boldsymbol{b}, y)$. But, if $s = \mathrm{qtp}_{\mathcal{M}}(\boldsymbol{a}, \boldsymbol{b})$, then this contradicts the 3-amalgamation of q, r, s. □

Strong universal duplication of quantifier-free types is important, as, in saturated models (of regular cardinal size) with quantifier-elimination, it implies indivisibility.

Proposition 17. Suppose

(1) $|M|$ is a regular cardinal,
(2) $\mathcal{M}$ is an $|M|$-saturated structure with quantifier-elimination, and
(3) $\mathcal{M}$ has strong universal duplication of quantifier-free types.

Then $\mathcal{M}$ is indivisible.

Proof. Let $|M| = \kappa$. Suppose, to get a contradiction, that $\mathcal{P}$ is a partition of M with $|\mathcal{P}| < \kappa$ such that no element of $\mathcal{P}$ contains a copy of $\mathcal{M}$ as a subset. Let $(P_i)_{i\in\zeta}$ be an enumeration of $\mathcal{P}$, where $|\mathcal{P}| = \zeta$. Suppose $(a_i)_{i\in\kappa}$ is an enumeration of $\mathcal{M}$.

We now define a sequence of tuples $(\boldsymbol{b}_i)_{i\in\zeta}$, a sequence of quantifier-free types $(r_i(\boldsymbol{z}_i, t_i))_{i\in\zeta}$, and a sequence of quantifier-free types $(u_i(\boldsymbol{v}_i, s_i))_{i\in\zeta}$. We do this by induction on $\lambda < \zeta$ and further require them to satisfy the following conditions:

(a) $|\boldsymbol{b}_\lambda| < \kappa$,
(b) $\boldsymbol{b}_\lambda \subseteq P_\lambda$,

(c) for $i < \lambda$ and each element of c in $\boldsymbol{b}_\lambda$, we have $\mathcal{M} \models r_i(\boldsymbol{b}_i, c)$,
(d) $\mathcal{M} \models (\forall y \in P_\lambda) \neg \bigwedge_{i \leq \lambda} r_i(\boldsymbol{b}_i, y)$,
(e) $\mathcal{M} \models (\forall \boldsymbol{v}_\lambda)(\forall y) \big(u_\lambda(\langle\!\langle \boldsymbol{v}_i \rangle\!\rangle_{i \in \lambda}, y) \rightarrow \bigwedge_{i \leq \lambda} r_i(\boldsymbol{v}_i, y)\big)$,
(f) $\mathcal{M} \models (\exists y)\, u_\lambda(\langle\!\langle \boldsymbol{b}_i \rangle\!\rangle_{i \in \lambda}, y)$.

For all $\lambda < \zeta$, let $D_\lambda = \{y \in \mathcal{M} : u_\lambda(\langle\!\langle \boldsymbol{b}_i \rangle\!\rangle_{i \in \lambda}, y)\}$.

Case 1: $\lambda = 0$. As P_0 does not contain an isomorphic copy of $\mathcal{M}$, there must be a tuple $\boldsymbol{b}_0 \in P_0$ with $|\boldsymbol{b}_0| < \kappa$ and a quantifier-free type $r_0(\boldsymbol{x}, y)$ such that $\mathcal{M} \models (\exists y) \bigwedge r_0(\boldsymbol{b}_0, y)$ but $\mathcal{M} \models (\forall y \in P_0) \neg \bigwedge r_0(\boldsymbol{b}_0, y)$. Let $u_0 = r_0$. Note $D_0 = \{y \in \mathcal{M} : \mathcal{M} \models \bigwedge u_0(\boldsymbol{b}_0, y)\} \neq \varnothing$.

Case 2: $\lambda = \alpha + 1$. By our inductive assumption (f), there must be an element $y \in D_\alpha$ and hence, by (e), a $y \in \mathcal{M}$ such that $\mathcal{M} \models \bigwedge_{i \leq \alpha} r_i(\boldsymbol{b}_i, y)$. Note, by the inductive hypothesis (d), for all $i \leq \alpha$, $P_i \cap D_\alpha = \varnothing$. We now break into two subcases.

Subcase 2.1: $D_\alpha \cap P_{\alpha+1} = \varnothing$. In this case let $\boldsymbol{b}_{\alpha+1}$ be the empty string and let $r_{\alpha+1}(y)$ be any unary quantifier-free type realized in $\mathcal{M}$. Further let $u_{\alpha+1} = u_\alpha$.

Subcase 2.2: $D_\alpha \cap P_{\alpha+1} \neq \varnothing$. As $\mathcal{M}$ has strong universal duplication of quantifier-free types, $\mathcal{M}\restriction_{D_\alpha} \cong \mathcal{M}$. But $P_{\alpha+1}$ does not contain a copy of $\mathcal{M}$. Therefore there must be a tuple $\boldsymbol{b}_{\alpha+1}$ of size $< \kappa$ in $P_{\alpha+1} \cap D_\alpha$ and a quantifier-free type $r_{\alpha+1}(\boldsymbol{b}_{\alpha+1}, y)$ that is realized in $\mathcal{M}\restriction_{D_\alpha}$, say by an element a, but not realized in $P_{\alpha+1}$. In particular, $\boldsymbol{b}_{\alpha+1}$ and $r_{\alpha+1}$ satisfy the inductive hypothesis. We therefore let $u_{\alpha+1}$ be the quantifier-free type of $\langle\!\langle \boldsymbol{b}_i \rangle\!\rangle_{i \in \alpha+1}{}^\wedge a$.

Case 3: $\lambda = \omega \cdot \delta$. Let $u^*(y) = \bigcup_{i \in \omega \cdot \delta} u_i(\langle\!\langle \boldsymbol{b}_j \rangle\!\rangle_{j \leq i}, y)$. Note $u^*(y)$ is finitely consistent, and so it is consistent. Further, as $|\boldsymbol{b}_i| < \kappa$, $\omega \cdot \delta < \kappa$ and κ is regular we have $\left|\bigcup_{i < \omega \cdot \delta} \boldsymbol{b}_i\right| < \kappa$. Therefore $\mathcal{M} \models (\exists y)\, u^*(y)$ as $\mathcal{M}$ is κ-saturated. But this implies that $\bigcap_{i < \omega \cdot \delta} D_i \neq \varnothing$. We now break into two subcases.

Subcase 3.1: $\big(\bigcap_{i < \omega \cdot \delta} D_i\big) \cap P_{\omega \cdot \delta} = \varnothing$. In this case let $\boldsymbol{b}_{\omega \cdot \delta}$ be the empty sequence and $r_{\omega \cdot \delta}(y)$ be any unary type realized in $\mathcal{M}$ and $u_{\omega \cdot \delta}(\boldsymbol{v}_{\omega \cdot \delta}, y) = \bigcup_{i \in \omega \cdot \delta} u_i(\boldsymbol{v}_i, y)$.

Subcase 3.2: $\big(\bigcap_{i < \omega \cdot \delta} D_i\big) \cap P_{\omega \cdot \delta} \neq \varnothing$. Note $\bigcap_{i < \omega \cdot \delta} D_i = \{y \in \mathcal{M} : \mathcal{M} \models u^*(y)\}$. So, as $\mathcal{M}$ has strong universal duplication of quantifier-free types, $\mathcal{M}\restriction_{\bigcap_{i < \omega \cdot \delta} D_i} \cong \mathcal{M}$. But $P_{\omega \cdot \delta}$ does not contain a copy of $\mathcal{M}$. Therefore there must be a tuple $\boldsymbol{b}_{\omega \cdot \delta}$ in $P_{\omega \cdot \delta} \cap \big(\bigcap_{i < \omega \cdot \delta} D_i\big)$ and a quantifier-free type $r_{\omega \cdot \delta}(\boldsymbol{b}_{\omega \cdot \delta}, y)$ that is realized in $\mathcal{M}\restriction_{\bigcap_{i < \omega \cdot \delta} D_i}$, by some tuple a, but not realized in $P_{\omega \cdot \delta}$. Let $u_{\omega \cdot \delta}$ be the quantifier-free type of $\langle\!\langle \boldsymbol{b}_i \rangle\!\rangle_{i \in \omega \cdot \delta + 1}{}^\wedge a$. In particular $\boldsymbol{b}_{\omega \cdot \delta}$, $r_{\omega \cdot \delta}$ and $u_{\omega \cdot \delta}$ satisfy the inductive hypothesis.

Finally, note that the set $\bigcup_{i \in \zeta} r_i(\boldsymbol{b}_i, y)$ is finitely consistent and $|\langle\!\langle \boldsymbol{b}_i \rangle\!\rangle_{i \in \zeta}| < \kappa$. Therefore, as $\mathcal{M}$ is κ-saturated, there is a $y \in \mathcal{M}$ such that $\mathcal{M} \models \bigwedge_{i \in \zeta} r_i(\boldsymbol{b}_i, y)$. But then for all $i \in \zeta$, $y \notin P_i$, contradicting the fact that $\mathcal{P}$ is a partition of $\mathcal{M}$. ☐

We now give a condition that will ensure being sunflowerable. This allows us to show that many well-known structures, such as the rational linear ordering and the Rado graph are sunflowerable.

Theorem 18. Suppose

(1) $\mathcal{M}$ is an $\mathscr{L}$-structure,
(2) $|M|$ is an infinite regular cardinal,
(3) $\mathcal{M}$ is indivisible and ultrahomogeneous, and
(4) $\mathcal{M}$ has universal duplication of quantifier-free types.

Then $\mathcal{M}$ is sunflowerable.

Proof. First note, by Lemmas 12 and 14, if $M_0 \subseteq M$ then $\mathcal{M}\restriction_{M_0}$ is a substructure of $\mathcal{M}$, i.e., $\mathcal{M}$ restricted to a subset always gives rise to a substructure.

Suppose $n \in \omega$, S is a set, $Y \subseteq \mathcal{P}_n(S)$ and $\mathcal{Y}$ is an $\mathscr{L}$-structure with underlying set Y and $\mathcal{Y} \cong \mathcal{M}$. As $|Y| = \kappa$ we can assume without loss of generality that $\bigcup Y \subseteq \kappa$ as well. We will now prove, by induction on n, that there is $Z \subseteq Y$ such that $\mathcal{Y}\restriction_Z \cong \mathcal{M}$ and Z is a sunflower.

Case 1: $n = 1$. In this case $y_0 \cap y_1 = \varnothing$ for all distinct $y_0, y_1 \in Y$ and hence Y is the desired sunflower.

Case 2: $n = k+1$. Suppose, to get a contradiction, that there is no $Z \subseteq Y$ that is a sunflower with $\mathcal{Y}\restriction_Z \cong \mathcal{M}$. For each $x \in \kappa$, let $Y_x^\circ = \{y \in Y : x \in y\}$. Let $Y_x = \{y \backslash \{x\} : y \in Y_x^\circ\}$. Note $Y_x \subseteq \mathcal{P}_k(S)$ and the map $u_x \colon Y_x \to Y_x^\circ$ with $u_x(y) = y \cup \{x\}$ is a bijection. We let $\mathcal{Y}_x$ be the $\mathscr{L}$-structure on Y_x such that for all atomic formulas ψ of arity ℓ and all $z_0, \ldots, z_{\ell-1} \in \mathcal{Y}_x$, $\mathcal{Y}_x \models \psi(z_0, \ldots, z_{\ell-1})$ if and only if $\mathcal{Y} \models \psi(u_x(z_0), \ldots, u_x(z_{\ell-1}))$.

If there is an $x \in \kappa$ and $C_x \subseteq Y_x$ is such that $\mathcal{Y}_x\restriction_{C_x} \cong \mathcal{M}$, then, by the inductive hypothesis applied to $\mathcal{Y}_x\restriction_{C_x}$, there is a $Z_x \subseteq C_x$ such that $\mathcal{Y}_x\restriction_{Z_x} \cong \mathcal{M}$ and Z_x is a sunflower. But then $u_x[Z_x] \subseteq Y$ is a sunflower and $\mathcal{Y}\restriction_{u_x[Z]} \cong \mathcal{M}$, contradicting our assumption. It therefore suffices to consider the case when, for all $x \in X$, Y_x contains no substructure isomorphic to $\mathcal{M}$.

Let $\boldsymbol{a} = (a_i)_{i\in\kappa}$ be an enumeration of $\mathcal{M}$. We define the sequence $(z_i)_{i\in\kappa} \subseteq Y$ by induction, where for all $\alpha \in \kappa$, $\mathrm{qtp}_{\mathcal{M}}(\langle z_i \rangle_{i\in\alpha}) = \mathrm{qtp}_{\mathcal{M}}(\langle a_i \rangle_{i\in\alpha})$, and for all $i < j \in \kappa$, $z_i \cap z_j = \varnothing$.

Let $z_0 = a_0$. Let $\alpha \in \kappa$ and assume z_i is defined for all $i \in \alpha$. Let $r_\alpha = \sup \bigcup_{j<\alpha} z_j$. Note $|z_i| = n$ for all $i \in \alpha$ and $|\alpha| < \kappa$, so $n \cdot \alpha < \kappa$, because κ is an infinite regular cardinal, and hence $r_\alpha < \kappa$. Let $C_\alpha^\circ = \left\{b \in \mathcal{Y} : \mathrm{qtp}_{\mathcal{Y}}(\langle z_i \rangle_{i\in\alpha}, b) = \mathrm{qtp}_{\mathcal{M}'}(\langle a_i \rangle_{i\in\alpha}, a_\alpha)\right\}$. Note $C_\alpha^\circ \neq \varnothing$, as $\mathcal{Y}$ is ultrahomogeneous. Let $C_\alpha \subseteq C_\alpha^\circ$ be such that $\mathcal{Y}\restriction_{C_\alpha} \cong \mathcal{Y}$. Note we can always find such a C_α as $\mathcal{Y}$ has universal duplication of quantifier-free types. Now suppose $C_\alpha \subseteq \bigcup_{i\le r_\alpha} Y_i^\circ$. Then, as $\mathcal{Y}$ is indivisible, there must be an $j \le r_\alpha$ and a $D_\alpha \subseteq C_\alpha \cap Y_j^\circ$ where $D_\alpha \cong \mathcal{Y}$. But then $u_j^{-1}[D_\alpha] \subseteq \mathcal{Y}_j$ and $u_j^{-1}[D_\alpha] \cong \mathcal{Y}$. Therefore, by the inductive hypothesis applied to $u_j^{-1}[D_\alpha]$, there is an $E_\alpha \subseteq u_j^{-1}[D_\alpha]$ that is a sunflower for which $\mathcal{Y}_j\restriction_{E_\alpha} \cong \mathcal{Y}$. Therefore $u_j[E_\alpha] \subseteq Y$ is a sunflower and $\mathcal{Y}\restriction_{u_j[E_\alpha]} \cong \mathcal{Y}$. But we assumed no such subset exists, therefore there must be some $z_\alpha \in C_\alpha \backslash \bigcup_{i\le r_\alpha} Y_i^\circ$. Hence for all $i \le r_\alpha$ we have $i \notin z_\alpha$ as $z_\alpha \notin Y_i$. Therefore, for all $j < \alpha$, $z_j \cap z_\alpha = \varnothing$.

Let $Z = \{z_i\}_{i\in\kappa}$. By the construction, the map $z_i \mapsto a_i$ is an isomorphism from $\mathcal{Y}\restriction_Z \to \mathcal{M}$. Further, as $z_i \cap z_j = \varnothing$ for all $i < j < \kappa$, we have that Z is a sunflower. But this contradicts our assumption that no such Z exist. $\square$

We also have the following corollaries.

Corollary 19. Suppose

(1) $\mathcal{M}$ is an $\mathscr{L}$-structure with $|M|$ a regular cardinal,
(2) $\mathcal{M}$ is an $|M|$-saturated structure with quantifier-elimination, and
(3) $\mathcal{M}$ has strong universal duplication of quantifier-free types.

Then $\mathcal{M}$ is sunflowerable.

Proof. This follows immediately from Proposition 17, Theorem 18 and the fact that saturated structures with quantifier-elimination are ultrahomogeneous. $\square$

Corollary 20. Suppose

(1) $\mathcal{M}$ is an $\mathscr{L}$-structure with $|M|$ regular,
(2) $\mathcal{M}$ is an $|M|$-saturated structure with quantifier-elimination,
(3) all elements of $\mathcal{M}$ have the same quantifier-free type, and
(4) $\mathcal{M}$ has the 3-disjoint amalgamation property (see Definition 40).

Then $\mathcal{M}$ is sunflowerable.

Proof. This follows immediately from Lemma 16 and Corollary 19. $\square$

We can apply Corollary 19 to give many examples of sunflowerable structures.

Lemma 21. If κ is an infinite cardinal and $(X, \le)$ is a κ-saturated model of the theory of dense linear orderings without endpoints with $|X| = \kappa$ then $(X, \le)$ is sunflowerable.

Proof. If the theory of dense linear orderings without endpoints has a κ-saturated model of size κ, then $\kappa = \kappa^{<\kappa}$. Hence κ is regular. As the theory of dense linear orderings without endpoints has quantifier-elimination, by Corollary 19, it suffices to show that $(X, \leq)$ has strong universal duplication of κ-quantifier-free types.

Suppose $\boldsymbol{a}$ is a sequence of elements of X of size $< \kappa$ and suppose $a \in X \backslash \boldsymbol{a}$. Let $q(x)$ be the type of a over $\boldsymbol{a}$ in X. Let $B_{\boldsymbol{a},a} = \{b : (X, \leq) \models q(b)\}$. We can then divide the elements of $\boldsymbol{a}$ into two disjoint sets $A_<$, $A_>$ where $b_< < a < b_>$ for all $b_< \in A_<$ and $b_> \in A_>$. Let

$$p(x, y, z) = \{b_< < x : b_< \in A_<\} \cup \{x < y, y < z\} \cup \{z < b_> : b_> \in A_>\}.$$

Note for any $c, d \in B_{\boldsymbol{a},a}$ with $c < d$ the type $p(c, y, d)$ is finitely consistent and hence realized. Therefore $(B_{\boldsymbol{a},a}, \leq)$ is a dense linear order and $|B_{\boldsymbol{a},a}| = |X| = \kappa$. Further, for all $c \in B_{\boldsymbol{a},a}$, the type $\{b_< < x : b_< \in A_<\} \cup \{x < c\}$ and the type $\{b_> > x : b_> \in A_>\} \cup \{x > c\}$ are consistent and hence realized. Therefore $(B_{\boldsymbol{a},a}, \leq)$ is a dense linear ordering without endpoints.

Now suppose $C \subseteq B_{\boldsymbol{a},a}$ with $|C| < \kappa$ and $r(x)$ is a type over C. If $r(x)$ is of the form $x = c$ for some $c \in C$, then $r(x)$ is realized in $B_{\boldsymbol{a},a}$. If $r(x)$ is not of the form $x = c$, we can partition C into two sets $C_<$ and $C_>$ such that $r(x)$ is equivalent to $\{c_< < x : c_< \in C_<\} \cup \{x < c_> : c_> \in C_>\}$. But then $q(x) \cup r(x)$ is finitely consistent, so there must be some d such that $(X, \leq) \models q(d) \cup r(d)$. But such a d must be in $B_{\boldsymbol{a},a}$, so $(B_{\boldsymbol{a},a}, \leq)$ is κ-saturated. Hence $(B_{\boldsymbol{a},a}, \leq) \cong (X, \leq)$. So, as $\boldsymbol{a}, a$ were arbitrary, $(X, \leq)$ has strong universal duplication of κ-quantifier-free types. $\square$

Lemma 22. If κ is a regular cardinal and (X, E) is a κ-saturated model of the theory of the Rado graph with $|X| = \kappa$, then (X, E) is sunflowerable.

Proof. This follows immediately from Corollary 20 and the facts that the theory of the Rado graph has quantifier-elimination and any saturated model of the theory of the Rado graph has the 3-disjoint amalgamation property. $\square$

Example 23. For $1 \leq \ell \leq k$, let K^k_ℓ be the collection of all k-uniform hypergraphs such that all ℓ-element subsets are in an edge. For $J \subseteq K^k_\ell$ let $\mathcal{H}_J$ be the collection of all k-uniform hypergraphs that do not realize any element of J. Note $\mathcal{H}_J$ is an age with (HP), (JEP), and (2-DAP), see Section 4.1. Further, if $\ell \geq 3$ then $\mathcal{H}_J$ has the 3-disjoint amalgamation property as well. In particular, there is a Fraïssé limit $\mathcal{G}_J$ of $\mathcal{H}_J$ and, by Corollary 20, $\mathcal{G}_J$ is sunflowerable.

Note that these techniques do not apply to all ultrahomogeneous structures, even if the structures have free-amalgamation, as the Rado graph does. For $n \geq 3$ let $\mathcal{H}_n$ be the countable generic K_n-free graph.

Lemma 24. The graph $\mathcal{H}_n$ does not have universal duplication of quantifier-free types.

Proof. Let $a_0, \dots, a_{n-2}$ be a complete subgraph of $\mathcal{H}_n$. If $\mathcal{H}_n$ had universal duplication of quantifier-free types, then there would have to be a set $X \subseteq \{b \in \mathcal{H}_n : \mathrm{qtp}_{\mathcal{H}_n}(a_0, \dots, a_{n-2}) = \mathrm{qtp}_{\mathcal{H}_n}(a_0, \dots, a_{n-3}, b)\}$ such that $\mathcal{H}_n\restriction_X \cong \mathcal{H}_n$. In particular, there would have to be two b_0, b_1 such that

$$\mathrm{qtp}_{\mathcal{H}_n}(a_0, \dots, a_{n-2}) = \mathrm{qtp}_{\mathcal{H}_n}(a_0, \dots, a_{n-3}, b_0) = \mathrm{qtp}_{\mathcal{H}_n}(a_0, \dots, a_{n-3}, b_1)$$

and $\mathcal{H}_n \models E(b_0, b_1)$. But then $\{a_i\}_{i \in [n-3]} \cup \{b_0\}$ is a complete K_{n-1} and $\{a_i\}_{i \in [n-3]} \cup \{b_1\}$ is a complete K_{n-1}. So $\{a_i\}_{i \in [n-3]} \cup \{b_0, b_1\}$ is a complete K_n contradicting the fact that $\mathcal{H}_n$ is K_n-free. $\square$

3 Linear ordered sunflowers

We now consider, for infinite cardinals κ, which linear orders of size κ are sunflowerable.

Definition 25. Suppose $\mathcal{Y} = (Y, \leq)$ is a linear ordering and for all $y \in Y$, $\mathcal{X}_y = (X_y, \leq)$ is a linear ordering. We define $\left(\sum_{y \in \mathcal{Y}} \mathcal{X}_y, \leq\right)$ to be the linear ordering where the underlying set is $\{(y, x) : y \in Y \text{ and } x \in X_y\}$ and $(y_0, x_0) \leq (y_1, x_1)$ if either $y_0 < y_1$ or $y_0 = y_1$ and $x_0 \leq x_1$. We call $\sum_{y \in \mathcal{Y}} \mathcal{X}_y$ a *lexicographical sum.*

Definition 26. A linear ordering $(X, \leq)$ is *κ-dense* if it has size at least 2 and for all distinct $a, b \in X$, $|\{c : a < c < b\}| = \kappa$.

Note that $(\mathbb{Q}, \leq)$ is ω-dense and the κ-dense linear orderings of size κ are κ-sized analogs of $(\mathbb{Q}, \leq)$. This motivates the following definition.

Definition 27. A linear ordering $(X, \leq)$ is *κ-scattered* if there does not exist a $Y \subseteq X$ such that $(Y, \leq)$ is κ-dense. We say a linear ordering is *scattered* if it is ω-scattered.

Hausdorff gave a characterization of scattered linear orderings, which was generalized in [1] to κ-scattered linear orderings.

Definition 28. Let $\mathfrak{BL}_\kappa$ be the collection of linear orderings that are either

(1) of size $< \kappa$,
(2) a well-ordering, or
(3) the inverse of a well-ordering.

Let $\mathfrak{L}_\kappa$ be the closure of $\mathfrak{BL}_\kappa$ under lexicographical sums ordered by elements of $\mathfrak{BL}_\kappa$.

Proposition 29 (Abraham et al., [1, Theorem 3.10])**.** The following are equivalent for a linear ordering $\mathcal{L}$:

(1) $\mathcal{L}$ is κ-scattered,
(2) $\mathcal{L} \in \mathfrak{L}_\kappa$.

Definition 30. Let $\mathfrak{D}_\kappa$ be the collection of linear orderings of size $< \kappa$. Consider the following collection of linear orderings defined by induction. First, $\mathfrak{L}_{0,\kappa}$ contains all isomorphic copies of elements of $\mathfrak{D}_\kappa$. If $\mathfrak{L}_{i,\kappa}$ has been defined for all $i \in \omega \cdot \delta$, let $\mathfrak{L}_{\omega\cdot\delta,\kappa} = \bigcup_{i \in \omega\cdot\delta} \mathfrak{L}_{i,\kappa}$. If $\mathfrak{L}_{\alpha,\kappa}$ has been defined and $\mathcal{X}$ is a linear ordering, let $\mathfrak{L}^{\mathcal{X}}_{\alpha,\kappa}$ be the collection of linear orderings of the form $\sum_{x \in X} \mathcal{A}_x$ where $\mathcal{A}_x \in \mathfrak{L}_{\alpha,\kappa}$ for all $x \in X$. Let $\mathfrak{L}_{\alpha+1,\kappa} = \bigcup_{\mathcal{Y} \in \mathfrak{D}_\kappa} \mathfrak{L}^{\mathcal{Y}}_{\alpha,\kappa} \cup \mathfrak{L}^{\kappa}_{\alpha,\kappa} \cup \mathfrak{L}^{\kappa^*}_{\alpha,\kappa}$. Let $\mathfrak{L}_{\infty,\kappa} = \bigcup_{i \in \mathrm{ORD}} \mathfrak{L}_{i,\kappa}$. For $\mathcal{A} \in \mathfrak{L}_{\infty,\kappa}$, let $r_\kappa(\mathcal{A})$ be the least α such that $\mathcal{A} \in \mathfrak{L}_{\alpha,\kappa}$. We call $r_\kappa(\mathcal{A})$ the *κ-scattered construction rank* of $\mathcal{A}$.

Lemma 31. $\mathfrak{L}_{\infty,\kappa}$ is the collection of κ-scattered linear orderings of size at most κ.

Proof. It is immediate that every element of $\mathfrak{L}_{\infty,\kappa}$ has size at most κ. Further, $\mathfrak{L}_{\infty,\kappa} \subseteq \mathfrak{L}_\kappa$ and so by Proposition 29 every element of $\mathfrak{L}_{\infty,\kappa}$ is κ-scattered. Also by Proposition 29, it suffices to show that for all ordinals β of size at most κ we have $\beta, \beta^* \in \mathfrak{L}_{\infty,\kappa}$. If $\beta \leq \kappa$ it is immediate that $\beta \in \mathfrak{L}_{\infty,\kappa}$.

Now assume, to get a contradiction, that for some $\beta < \kappa^+$ we have $\beta \notin \mathfrak{L}_{\infty,\kappa}$ and let β be minimal as such. If $\beta = \alpha + 1$, then $\alpha \in \mathfrak{L}_{\infty,\kappa}$ and $\alpha + S_1 = \beta$ where S_1 is the 1-point linear ordering (and hence in $\mathfrak{D}_\kappa$). Therefore $\beta \in \mathfrak{L}_{\infty,\kappa}$, getting us a contradiction. If β is a limit, then there must be an increasing sequence of ordinals $(\zeta_i)_{i\in\eta}$ (for $\eta \leq \kappa$) such that $\beta = \sum_{i\in\eta} \zeta_i$. But, as each $\zeta_i \in \mathfrak{L}_{\infty,\kappa}$, we have $\beta \in \mathfrak{L}_{\infty,\kappa}$. This gets us our contradiction, ensuring that for all $\beta < \kappa^+$, $\beta \in \mathfrak{L}_{\infty,\kappa}$.

The case that $\beta^* \in \mathfrak{L}_{\infty,\kappa}$ is identical. □

We now give an important property of κ-scattered linear orderings of size κ.

Proposition 32. Suppose $\mathcal{X}$ is a κ-scattered linear ordering of size κ. Then one of three things happens.

(1) We can find a linear ordering $\mathcal{D}$ with $|D| < \kappa$ and κ-scattered linear orderings $(\mathcal{X}_i)_{i\in D}$ such that $\mathcal{X} = \sum_{i\in\mathcal{D}} \mathcal{X}_i$ and for all $i \in D$, $\mathcal{X}_i$ does not contain an isomorphic copy of $\mathcal{X}$ as a subset,
(2) We can find κ-scattered linear orderings $(\mathcal{X}_i)_{i\in\kappa}$ such that $\mathcal{X} = \sum_{i\in\kappa} \mathcal{X}_i$ and for all $i \in \kappa$, $\mathcal{X}_i$ does not contain a copy of $\mathcal{X}$,
(3) We can find κ-scattered linear orderings $(\mathcal{X}_i)_{i\in\kappa}$ such that $\mathcal{X} = \sum_{i\in\kappa^*} \mathcal{X}_i$ and for all $i \in \kappa$, $\mathcal{X}_i$ does not contain a copy of $\mathcal{X}$.

Proof. Let $\boldsymbol{\mathcal{Y}}$ be the collection of κ-scattered linear orderings of size κ which $\mathcal{X}$ embeds into. Let $\mathcal{Z}$ be an element of $\boldsymbol{\mathcal{Y}}$ with $r_\kappa(\mathcal{Z})$ minimal. Let $\tau\colon \mathcal{X} \to \mathcal{Z}$ be an embedding. Note $r_\kappa(\mathcal{Z}) = \alpha + 1$ for some ordinal α as no κ-sized linear ordering embeds into an element of $\mathfrak{D}_\kappa = \mathfrak{L}_{0,\kappa}$ and $\mathcal{Z}$ has minimal κ-scattered construction rank among κ-scattered linear orderings containing a copy of $\mathcal{X}$. We now have three cases depending on the form of $\mathcal{Z}$.

Case 1: $\mathcal{Z} \in \mathfrak{L}^{Y}_{\alpha,\kappa}$ and $\mathcal{Y} \in \mathfrak{D}_\kappa$. There are then $(\mathcal{A}_i)_{i\in Y} \subseteq \mathfrak{L}_{\alpha,\kappa}$ such that $\mathcal{Z} = \sum_{i\in Y} \mathcal{A}_i$. Let $\mathcal{X}_i = \tau^{-1}[\mathcal{A}_i]$. Note that if $\mathcal{X}$ embedded into $\mathcal{X}_i$ then $\mathcal{X}$ would embed into $\mathcal{A}_i$ contradicting the minimality of $r_\kappa(\mathcal{Z})$. Therefore $\mathcal{X}_i$ satisfies condition (1) for all $i \in Y$.

Case 2: $\mathcal{Z} \in \mathfrak{L}^{\kappa}_{\alpha,\kappa}$. There are then $(\mathcal{A}_i)_{i\in \kappa} \subseteq \mathfrak{L}_{\alpha,\kappa}$ such that $\mathcal{Z} = \sum_{i\in \kappa} \mathcal{A}_i$. For $i \in \kappa$ let $\mathcal{X}_i = \tau^{-1}[\mathcal{A}_i]$. Note that if $\mathcal{X}$ embedded into $\mathcal{X}_i$ then $\mathcal{X}$ would embed into $\mathcal{A}_i$ contradicting the minimality of $r_\kappa(\mathcal{Z})$. Therefore $(\mathcal{X}_i)_{i\in\kappa}$ satisfy condition (2).

Case 3: $\mathcal{Z} \in \mathfrak{L}^{\kappa^*}_{\alpha,\kappa}$. There are then $(\mathcal{A}_i)_{i\in \kappa} \subseteq \mathfrak{L}_{\alpha,\kappa}$ such that $\mathcal{Z} = \sum_{i\in \kappa^*} \mathcal{A}_i$. For $i \in \omega$, let $\mathcal{X}_i = \tau^{-1}[\mathcal{A}_i]$. Note that if $\mathcal{X}$ embedded into $\mathcal{X}_i$, then $\mathcal{X}$ would embed into $\mathcal{A}_i$ contradicting the minimality of $r_\kappa(\mathcal{Z})$. Therefore $(\mathcal{X}_i)_{i\in\kappa}$ satisfy condition (3). ☐

We therefore have the following theorem.

Theorem 33. Suppose $\mathcal{X} = (X, \leq)$ is a κ-scattered linear ordering of size κ. Then the following are equivalent:

(1) $\mathcal{X}$ is 2-sunflowerable,
(2) $\mathcal{X}$ is sunflowerable,
(3) $\mathcal{X}$ has order type κ or κ^*.

Proof. It is immediate that (2) implies (1). Suppose (3) holds. First note that any subset of κ of size κ has order type κ and any subset of κ^* of size κ has order type κ^*. Therefore both κ and κ^* are sunflowerable by Theorem 1. Hence (3) implies (2).

Now suppose (3) fails. By Proposition 32, there is a linear order $\mathcal{Y}$ such that either $|Y| < \kappa$ or the order type of $\mathcal{Y}$ is κ or κ^* and there is a collection $(\mathcal{X}_i)_{i\in Y}$ such that $\mathcal{X} = \sum_{i\in\mathcal{Y}} \mathcal{X}_i$ and $\mathcal{X}$ does not embed into any $\mathcal{X}_i$. Let $(X_i, \leq) = \mathcal{X}_i$. We can assume, without loss of generality, that $X_i \cap X_j = \varnothing$ for all distinct $i, j \in Y$ and $X_i \cap Y = \varnothing$ for all $i \in Y$. Let $Z = Y \cup (\bigcup_{i\in Y} X_i)$. Note $|Z| = |X| = \kappa$. Let $A \subseteq \mathcal{P}_2(Z)$ be the set $\{\{i, x\} : i \in \kappa, x \in X_i\}$. Say that $\{i, x\} \leq_A \{j, y\}$ if and only if either $i < j$ or $i = j$ and $x \leq y$. We then have $(A, \leq_A) \cong (X, \leq)$.

Suppose, to get a contradiction, that $A_0 \subseteq A$ is a sunflower with $(A_0, \leq_A) \cong \mathcal{X}$. Suppose $\{i, x\}, \{j, y\} \in A_0$ with $i, j \in Y$. If $\{i, x\} \cap \{j, y\} \neq \varnothing$ then $i = j$ and $A_0 \subseteq \{\{i, x\} : x \in X_i\}$. But $\mathcal{X} \cong (A_0, \leq_A)$ and so $\mathcal{X}$ embeds into $\mathcal{X}_i$, getting us our contradiction. Otherwise we must have $\{i, x\} \cap \{j, y\} = \varnothing$. In particular, for all distinct $\{i, x\}, \{j, y\} \in A_0$ we have $i \neq j$. Therefore the order type of $(A_0, \leq_A)$ must be a suborder type of Y. But then the order type of $(A_0, \leq_A)$ is either size $< \kappa$ or of order type κ or κ^*. This implies $\mathcal{X}$ is either of size $< \kappa$ or of ordertype κ or κ^*, contradicting our assumption on $\mathcal{X}$. Therefore $\mathcal{X}$ is not 2-sunflowerable and (1) fails. Hence (1) implies (3). ☐

In the case when there is a saturated model $\mathbb{Q}_\kappa$ of size κ of the theory of dense linear orderings, we have a weakening of the notion of scattered.

Definition 34. Suppose $\kappa = \kappa^{<\kappa}$. Let $\mathbb{Q}_\kappa$ be the unique (up to isomorphism) saturated model of the theory of dense linear orderings without endpoints of size κ. We say a linear ordering $(X, \leq)$ is $\mathbb{Q}_\kappa$-*scattered* if there does not exist a subset $Y \subseteq X$ such that $(Y, \leq) \cong \mathbb{Q}_\kappa$.

The following is immediate by Lemma 21 and the fact that all linear orders of size κ embed into $\mathbb{Q}_\kappa$.

Proposition 35. If $\kappa = \kappa^{<\kappa}$, $|X| = \kappa$ and $\mathcal{X}$ is not $\mathbb{Q}_\kappa$-scattered then $\mathcal{X}$ is sunflowerable.

When $\kappa = \omega$, there is a unique ω-dense linear ordering (up to isomorphism), i.e., $(\mathbb{Q}, \leq)$. Further, being $\mathbb{Q}$-scattered is the same as being scattered. Therefore we have the following characterization of sunflowerable countable linear orderings.

Theorem 36. Suppose $\mathcal{X}$ is a countably infinite linear ordering. Then the following are equivalent:

(1) $\mathcal{X}$ is 2-sunflowerable,
(2) $\mathcal{X}$ is sunflowerable,
(3) One of the following is true of $\mathcal{X}$:
  (a) $\mathcal{X}$ is isomorphic to $(\omega, \leq)$,
  (b) $\mathcal{X}$ is isomorphic to $(\omega^*, \leq)$, or
  (c) $(\mathbb{Q}, \leq)$ embeds into $\mathcal{X}$.

Proof. This is immediate from Lemma 21, Theorem 33, the fact that any non-scattered linear ordering embeds $\mathbb{Q}$, and that all countable linear orderings embed into $\mathbb{Q}$. □

Note any κ-dense but non-sunflowerable linear ordering would have to become ω-dense in any larger model of set theory that collapses κ to ω. Therefore every κ-dense linear ordering, whether or not it is sunflowerable, is sunflowerable in a forcing extension of the background model of set theory. We observe that being sunflowerable is not absolute.

Example 37. Let $\mathcal{X} = \omega_1 \times \mathbb{Q}$ ordered lexographically. $\mathcal{X}$ is ω_1-scattered and hence, by Theorem 33, not sunflowerable. But $\mathcal{X}$ is a model of the theory of dense linear orderings without endpoints. Therefore in any larger model of set theory that collapses ω_1 to ω we have that $\mathcal{X}$ is isomorphic to $\mathbb{Q}$ and hence is sunflowerable.

4 Sunflower property of an age

It is well known that, via a compactness argument, the finite Ramsey theorem follows from the infinite Ramsey theorem. A similar argument shows that the sunflower lemma (without bounds) follows from the fact that all infinite sets of finite tuples of the same size contain an infinite sunflower. Further, this argument can be extended to sunflowerable structures and gives rise to the notion of an age having the sunflower property. In this section we will make this precise and also give concrete upper bounds when the age has the 3-disjoint amalgamation property.

Remark 38. For the rest of this section $\mathscr{L}$ will be a countable relational language.

4.1 Properties of ages

We first recall various properties that will be important.

Definition 39. An *$\mathscr{L}$-age*, $\mathbf{K}$, is a collection of finite structures closed under isomorphism. If $\mathcal{M}$ is a structure then the *age of* $\mathcal{M}$ is the collection of structures isomorphic to a finite substructure of $\mathcal{M}$.

Definition 40. Suppose $\mathbf{K}$ is an $\mathscr{L}$-age.

(1) $\mathbf{K}$ has the *hereditary property*, (HP), if whenever $\mathcal{A} \in \mathbf{K}$ and $A_0 \subseteq A$ then $\mathcal{A}{\restriction}_{A_0} \in \mathbf{K}$.
(2) $\mathbf{K}$ has the *joint embedding property*, (JEP), if whenever $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ there is a $\mathcal{C} \in \mathbf{K}$ and embeddings $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$ and $i_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$.
(3) $\mathbf{K}$ has the *n-disjoint amalgamation property* (n-DAP), if, whenever $\mathcal{A}_0, \dots, \mathcal{A}_{n-1} \in \mathbf{K}$ and for all $i, j \in [n]$ we have $\mathcal{A}_i{\restriction}_{A_i \cap A_j} = \mathcal{A}_j{\restriction}_{A_i \cap A_j}$, then there is a $\mathcal{B} \in \mathbf{K}$ such that $\mathcal{B}{\restriction}_{A_i} = \mathcal{A}_i$ for all $i \in [n]$.

Note that the collection of n-disjoint amalgamation properties forms a hierarchy. In the literature there are two distinct ways to define this hierarchy of (n-DAP), see for example [14] (which matches Definition 40) and also the notion of *free* in [2] and n-DAP in [6] (which both match the notion of *basic n-disjoint amalgamation* in [14]). However, the most common uses of these notions occur on ages that have n-DAP for all $n \in \omega$. Any such age will have n-DAP for all n under one notion if and only if it has n-DAP for all n under the other notion (see, for example, [14, Proposition 3.6]).

Note a countable structure $\mathcal{M}$ has 3-amalgamation of quantifier-free types (Definition 15) if and only if its age has (3-DAP). The following is immediate.

Lemma 41. Suppose $\mathbf{K}$ has (3-DAP), $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ and $\mathcal{A}\restriction_{A\cap B} = \mathcal{B}\restriction_{A\cap B}$, then there is a $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{C}\restriction_A = \mathcal{A}$ and $\mathcal{C}\restriction_B = \mathcal{B}$.

We call such a $\mathcal{C}$ in Lemma 41 a *disjoint amalgamation* of $\mathcal{A}$ and $\mathcal{B}$. The following is also immediate.

Lemma 42. If $\mathbf{K}$ has (3-DAP), then $\mathbf{K}$ has (JEP).

As we will see later, we will need the following property to hold in any age we care about.

Definition 43. We say an age $\mathbf{K}$ in $\mathscr{L}$ has *unique unary quantifier-free types* if for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ and $a \in A$ and $b \in B$, we have $\mathrm{qtp}_{\mathcal{A}}(a) = \mathrm{qtp}_{\mathcal{B}}(b)$.

If $\mathbf{K}$ has (HP), then having unique unary quantifier-free types is equivalent to all structures on the one element set being isomorphic. The following lemma will be our main use of (3-DAP).

Lemma 44. Suppose

(1) $\mathbf{K}$ has (3-DAP),
(2) $\mathbf{K}$ has unique unary types,
(3) $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ with $A \cap B = \varnothing$, and
(4) $(a_i)_{i\in[n+1]}$ is an enumeration of $\mathcal{A}$.

Then there is a $\mathcal{C} \in \mathbf{K}$ where

(5) $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{C}$, and
(6) for all $b \in \mathcal{B}$, $\mathrm{qtp}_{\mathcal{C}}\big((a_i)_{i\in[n+1]}\big) = \mathrm{qtp}_{\mathcal{C}}\big((a_i)_{i\in[n]}{}^\wedge b\big)$.

Proof. Let $\{b_i\}_{i\in[\ell]}$ be an enumeration of $\mathcal{B}$. We define a sequence of structures $\mathcal{C}_i$ for $i \in [\ell] \cup \{-1\}$ as follows. Let $\mathcal{C}_{-1} = \mathcal{A}$. If $\mathcal{C}_k$ is defined then we let $\mathcal{C}_{k+1} \in \mathbf{K}$ be a structure where

(a) $C_{k+1} = C_k \cup \{b_{k+1}\}$,
(b) $\mathcal{C}_{k+1}\restriction_{C_k} = \mathcal{C}_k$,
(c) $\mathcal{C}_{k+1}\restriction_{\{b_r\}_{r\in[k+2]}} = \mathcal{B}\restriction_{\{b_r\}_{r\in[k+2]}}$, and
(d) $\mathrm{qtp}_{\mathcal{C}_{k+1}}\big((a_i)_{i\in[n+1]}\big) = \mathrm{qtp}_{\mathcal{C}_{k+1}}\big((a_i)_{i\in[n]}{}^\wedge b_{k+1}\big)$.

Note that the structures $\mathcal{C}_k$ and $\mathcal{B}\restriction_{\{b_r\}_{r\in[k+2]}}$ agree on their common intersection of $B\restriction_{\{b_r\}_{r\in[k+1]}}$.

Similarly, if $\mathcal{D}$ is the structure with underlying set $\{a_i\}_{i\in[n]}\cup\{b_{k+1}\}$ such that $\mathcal{D} \models \mathrm{qtp}_{\mathcal{C}_{k+1}}\big((a_i)_{i\in[n]}{}^\wedge b_{k+1}\big) = \mathrm{qtp}_{\mathcal{C}_{k+1}}\big((a_i)_{i\in[n+1]}\big)$, then $\mathcal{D}$ and $\mathcal{A}$ agree on their common intersection, which is $\{a_i\}_{i\in[n]}$. Finally note that $\mathcal{D}$ and $\mathcal{B}\restriction_{\{b_r\}_{r\in[k+2]}}$ agree on their common intersection of $\{b_{k+1}\}$ as $\mathbf{K}$ has unique unary types. We can therefore apply (3-DAP) to find such a $\mathcal{C}_{k+1} \in \mathbf{K}$. Finally let $\mathcal{C} = \mathcal{C}_{\ell-1}$. □

We now recall a couple important properties of ages.

Definition 45. We say an age $\mathbf{K}$ is *indivisible* at $\mathcal{A} \in \mathbf{K}$ if for all n there is a $\mathcal{B} \in \mathbf{K}$ such that whenever $\alpha\colon B \to [n+1]$ is a map, then there is an $\mathcal{A}^+ \subseteq \mathcal{B}$ where $\mathcal{A} \cong \mathcal{A}^+$ and $|\alpha[A^+]| = 1$. We call $\mathcal{B}$ a *witness to indivisibility* of $\mathcal{A}$ in $\mathbf{K}$ of order n.

When an age $\mathbf{K}$ has unique unary quantifier-free types and (HP), then there is, up to isomorphism, a unique element $\mathcal{A}_1$ of size 1. In this situation, being indivisible at all $\mathcal{B} \in \mathbf{K}$ is the same as a class having the $\mathcal{A}_1$-Ramsey property (see [15, § 5] for a definition of the Ramsey property for an age).

The following lemma is also immediate.

Lemma 46. If $\mathbf{K}$ has (JEP) and is indivisible, then $\mathbf{K}$ has unique unary quantifier-free types.

Proof. Suppose $\mathbf{K}$ has (JEP) but does not have unique unary quantifier-free types. As $\mathbf{K}$ has (JEP) there must be an $\mathcal{A} \in \mathbf{K}$ and $a_0, a_1 \in \mathcal{A}$ with $\mathrm{qtp}_{\mathcal{A}}(a_0) \neq \mathrm{qtp}_{\mathcal{A}}(a_1)$. Now suppose $\mathcal{B} \in \mathbf{K}$ is any structure containing an isomorphic copy of $\mathcal{A}$. Let $\alpha\colon B \to [2]$ be the map where $\alpha(b) = 0$ if and only if $\mathrm{qtp}_{\mathcal{B}}(b) \neq \mathrm{qtp}_{\mathcal{A}}(a_0)$. Then there can be no subset $B_0 \subseteq B$ with $|\alpha[B_0]| = 1$ and $\mathcal{B}\restriction_{B_0} \cong \mathcal{A}$. Hence $\mathbf{K}$ is not indivisible. □

The following definition will be important for determining bounds on the existence of structures whose underlying set is a sunflower.

Definition 47. Let $\beta\colon \omega \times \omega \to \omega \cup \{\infty\}$. We say β is a *bound on indivisibility* if β is non-decreasing in its first coordinate and if for all $\mathcal{A} \in \mathbf{K}$ and all $n \in \omega$ either

(1) $\mathbf{K}$ is not indivisible at $\mathcal{A}$ and $\beta(|A|, n) = \infty$, or
(2) there is a $\mathcal{B} \in \mathbf{K}$ with $|B| \leq \beta(|A|, n)$ and $\mathcal{B}$ witnessing the indivisibility of $\mathcal{A}$ in $\mathbf{K}$ of order n.

We will want to use known results about indivisibility, along with Theorem 56, to provide several examples of sunflowerable ages. However, as indivisibility is most often studied in infinite ultrahomogeneous structures, we present a result connecting the indivisibility of such a structure with the indivisibility of its age.

Lemma 48. Suppose $\mathcal{M}$ is an indivisible countable structure and $\mathbf{K}$ is the age of $\mathcal{M}$. Then $\mathbf{K}$ is indivisible.

Proof. Suppose to get a contradiction that $\mathcal{M}$ is indivisible but $\mathbf{K}$ is not. There must then be an $n \in \omega$ and an $\mathcal{A} \in \mathbf{K}$ such that for all $\mathcal{B} \in \mathbf{K}$ there is an $\alpha\colon \mathcal{B} \to [n+1]$ such that $\alpha^{-1}(i)$ does not contain a copy of $\mathcal{A}$ for any $i \in [n+1]$. Call such a map α a *bad map*. For each $\mathcal{B} \in \mathbf{K}$ let $\mathrm{Bad}(\mathcal{B})$ be the collection of bad maps with range $[n+1]$ and domain $\mathcal{B}$. Note for any $\mathcal{B}$, $\mathrm{Bad}(\mathcal{B})$ is finite. Let $(a_i)_{i\in\omega}$ be an enumeration of $\mathcal{M}$ without repetitions where $\mathcal{A} \cong \{a_i\}_{i<|\mathcal{A}|}$. For $k \in \omega$ let $\mathcal{B}_k = \mathcal{M}\restriction_{\{a_i\}_{|\mathcal{A}|+k}}$. Let $\mathrm{Bad} = \bigcup_{k\in\omega} \mathrm{Bad}(\mathcal{B}_k)$. Then $(\mathrm{Bad}, \subseteq)$ is a finitely branching infinite tree, therefore by Kőnig's lemma there must be an infinite branch $(\alpha_k)_{i\in\omega} \subseteq \mathrm{Bad}$. If $\alpha = \bigcup_{k\in\omega} \alpha_k$ we have $\alpha\colon \mathcal{M} \to [n+1]$ is a map such that $\alpha^{-1}(i)$ does not contain a copy of $\mathcal{A}$ for any $i \in [n+1]$. But this contradicts the indivisibility of $\mathcal{M}$. □

4.2 Sunflower property

We now introduce the notion of the sunflower property.

Definition 49. We say an age $\mathbf{K}$ has the *n-sunflower property* if for every $\mathcal{A} \in \mathbf{K}$ there is a $\mathcal{B} \in \mathbf{K}$ such that whenever S is a set and $f\colon B \to \mathcal{P}_n(S)$ then there is a $A' \subseteq B$ such that $\mathcal{A} \cong \mathcal{B}\restriction_{A'}$ and $f[A']$ is a sunflower. We say $\mathcal{B}$ *witnesses* that $\mathbf{K}$ has the n-sunflower property at $\mathcal{A}$.

If $\mathbf{K}$ has the n-sunflower property for all n then we say it has the *sunflower property*.

Because all elements of $\mathbf{K}$ are finite, note that for each set S and each $\mathcal{A} \in \mathbf{K}$ with $A \subseteq \mathcal{P}_n(S)$ we have $|\bigcup A| < \omega$. Therefore there is an $\mathcal{A}' \in \mathbf{K}$ with $A' \subseteq \mathcal{P}_n(\omega)$ and an injective map $\alpha\colon \bigcup A \to \omega$ such that if $\alpha'\colon A \to \mathcal{P}_n(\omega)$ is the map where for all $a \in A$, $\alpha'(a) = \{\alpha(x) : x \in a\}$, then α' is an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$. Therefore, if we so choose, we could assume without loss of generality that all elements of $\mathbf{K}$ in Definition 49 have underlying sets contained in $\mathcal{P}_n(\omega)$.

Example 50. Let $\mathbf{K}_{\mathrm{LO}}$ be the class of linear orders. If $\mathcal{A}, \mathcal{B} \in \mathbf{K}_{\mathrm{LO}}$ and $\mathcal{A}$ is of size k, then any subset of $\mathcal{B}$ of size k is isomorphic to $\mathcal{A}$. Therefore, by the Theorem 1, $\mathbf{K}_{\mathrm{LO}}$ has the sunflower property.

We now show the sunflower properties form a hierarchy.

Lemma 51. If $n \in \omega$ and $\mathbf{K}$ has the $(n+2)$-sunflower property, then $\mathbf{K}$ has the $(n+1)$-sunflower property.

Proof. Suppose $\mathcal{A} \in \mathbf{K}$ and $\mathcal{B}$ witnesses that $\mathbf{K}$ has the $(n+2)$-sunflower property at $\mathcal{A}$. Suppose S is a set and $f\colon B \to \mathcal{P}_{n+1}(S)$. Let $\mathcal{B}'$ be the unique structure on $f[B]$ such that f is an isomorphism. Let $X = \bigcup f[B]$ and $x \notin X$. Let $\alpha\colon \mathcal{P}_{n+1}(X) \to \mathcal{P}_{n+2}(X \cup \{x\})$ be the map where $\alpha(y) = y \cup \{x\}$. Then α is injective.

Let $\mathcal{B}^* = \alpha[\mathcal{B}']$. There must then be a subset $\mathcal{A}^* \subseteq \mathcal{B}^*$ such that $\mathcal{A}^*$ is a sunflower and $\mathcal{A}^* \cong \mathcal{A}$. Let $\mathcal{A}' = \alpha^{-1}[\mathcal{A}^*]$. We then have $\mathcal{A}' \cong \mathcal{A}$. Also, there is a d such that for distinct $a, b \in \mathcal{A}^*$ we have $d = a \cap b$. Further, we have have $x \in d$. Therefore $\alpha^{-1}(a) \cap \alpha^{-1}(b) = d \backslash \{x\}$ for distinct a, b. Hence, as $\alpha\restriction_{\mathcal{A}'}$ is an isomorphism from $\mathcal{A}'$ to $\mathcal{A}^*$, the underlying set of $\mathcal{A}'$ is a sunflower. ◻

One of the main reasons why we restricted attention to ages that had unique unary quantifier-free types is that without this property the ages, in general, do not have even the 2-sunflower property.

Lemma 52. Suppose

(1) $\mathbf{K}$ does not have unique unary quantifier-free types,
(2) $\mathbf{K}$ has (JEP), and
(3) $\mathbf{K}$ has at least one structure of size at least 3.

Then $\mathbf{K}$ does not have the 2-sunflower property.

Proof. Suppose $\mathbf{K}$ does have the 2-sunflower property to get a contradiction. By our assumptions on $\mathbf{K}$ there must be an $\mathcal{A} \in \mathbf{K}$ with $|A| \geq 3$ along with $a, b \in A$ such that $\mathrm{qtp}(a) \neq \mathrm{qtp}(b)$. Then there is a $\mathcal{B}$ witnessing that $\mathbf{K}$ has the 2-sunflower property at $\mathcal{A}$. Pick $a \in \mathcal{A}$ and let E be the collection of elements $c \in B$ such that $\mathrm{qtp}(c) = \mathrm{qtp}(a)$. Let B be the underlying set of $\mathcal{B}$. Let $Y = \{(0, b) : b \in E\} \cup \{(1, b) : b \in B \backslash E\}$. Let $\mathcal{B}'$ be the structure with underlying set Y such that $\pi_1\colon Y \to B$, given by projection onto the second coordinate, is an isomorphism. Then every sunflower contained in Y either has size ≤ 2, is contained in E, or is contained in $B \backslash E$. Therefore, as $|\mathcal{B}| \geq |\mathcal{A}| \geq 3$, no subset of $\mathcal{B}'$ is isomorphic to $\mathcal{B}$ and is a sunflower. ◻

4.3 Sunflower property from sunflowerable structures

We now show that the age of a countable sunflowerable structure has the sunflower property. This will allow us to give several examples of ages with the sunflower property.

Theorem 53. Suppose $\mathbf{K}$ is an age and $n \in \omega$. Further suppose $\mathcal{M}$ is a countable $\mathscr{L}$-structure where the age of $\mathcal{M}$ is $\mathbf{K}$ and $\mathcal{M}$ is n-sunflowerable. Then $\mathbf{K}$ has the n-sunflower property.

Proof. As there is a structure whose age is $\mathbf{K}$, $\mathbf{K}$ satisfies (HP) and (JEP). Suppose, to get a contradiction, that $\mathbf{K}$ does not have the n-sunflower property. Then there is some $\mathcal{A} \in \mathbf{K}$ such that for every $\mathcal{B} \in \mathbf{K}$ there is a set S and an injection $f\colon B \to \mathcal{P}_n(S)$ where, whenever $A' \subseteq B$ with $\mathcal{B}\restriction_{A'} \cong \mathcal{A}$, we have that A' is not a sunflower. Let $(c_i)_{i\in\omega}$ be an enumeration of $\mathcal{M}$. For $\ell \in \omega$ let $\mathcal{C}_\ell$ be the substructure of $\mathcal{M}$ where $C_\ell = \{c_i\}_{i\in[\ell]}$. Note, as the age of $\mathcal{M}$ is $\mathbf{K}$, we have $\mathcal{C}_\ell \in \mathbf{K}$.

If X is a set, let D^X_ℓ be the collection of structures $\mathcal{C}'_\ell$ such that $\mathcal{C}'_\ell \cong \mathcal{C}_\ell$, $\mathcal{C}'_\ell \subseteq \mathcal{P}_n(X)$ and no subset of $\mathcal{C}'_\ell$ isomorphic to $\mathcal{A}$ is a sunflower. Note for each set X and each $\mathcal{E} \in \mathcal{D}^X_\ell$, we can find a subset $X_0 \subseteq [n \cdot \ell]$ and a bijection $\zeta\colon \bigcup E \to X_0$ and an element $\mathcal{E}_0 \in \mathcal{D}^{[n\cdot\ell]}_\ell$ such that, if $\zeta^*\colon \mathcal{P}(\bigcup E) \to \mathcal{P}(X_0)$ is the corresponding bijection, then $\zeta^*\colon \mathcal{E} \to \mathcal{E}_0$ is an isomorphism of structures. In particular, every element of D^X_ℓ is isomorphic to an element of $D^{[n\cdot\ell]}_\ell$ via an isomorphism that preserves the structure of the underlying set. Therefore, for all $\ell \in \omega$, if $D_\ell = D^{[n\cdot\ell]}_\ell$, then we have $D_\ell \neq \varnothing$.

For $\ell \in \omega$, we will define a tree $(T_\ell, \subseteq)$ where the kth level of the tree is a set $F_{\ell,k}$, which we will define by induction. Let $F_{\ell,0} = D_\ell$. If $\mathcal{E} \in F_{\ell,k}$, let $F^*(\mathcal{E})$ be the collection of $\mathcal{G} \in D_{\ell+k+1}$ such that $\mathcal{E} \subseteq \mathcal{G}$. We let $F_{\ell,k+1} = \bigcup\{F^*(\mathcal{E}) : \mathcal{E} \in F_{\ell,k}\}$. Note that if $\mathcal{E} \in D_{\ell+k}$, then by permuting the elements $[(\ell + k) \cdot n]$ we can find an isomorphic copy of $\mathcal{E}^* \cong \mathcal{E}$ with $\mathcal{E}^* \in F_{\ell,k}$. Therefore, as $D_{\ell+k} \neq \varnothing$ for all $k \in \omega$, we have $F_{\ell,k} \neq \varnothing$

for all $k \in \omega$. But then T_ℓ is infinite. For all $k \in \omega$, as $D_{\ell+k}$ is finite, $F_{\ell,k}$ is finite as well. Therefore $(T_\ell, \subseteq)$ is finitely branching. Hence, by Kőnig's lemma, we must have an infinite branch $(\mathcal{E}_i)_{i\in\omega}$ in T_ℓ. Note that for $k \in \omega$, $|E_{k+1} \backslash E_k| = 1$. Let $\mathcal{E} = \bigcup_{i\in\omega} \mathcal{E}_i$. We then have $\mathcal{E} \cong \mathcal{M}$, where the unique element of $E_{k+1}\backslash E_k$ gets mapped to $c_{\ell+k+1}$.

Because $\mathcal{M}$ is n-sunflowerable, there must be an infinite sunflower $E^- \subseteq E$ such that $\iota\colon \mathcal{E}^- \cong \mathcal{M}$. As the age of $\mathcal{M}$ is $\mathbf{K}$, there must be a subset $A^- \subseteq E^-$ with $\mathcal{A}^- \cong \mathcal{A}$. In particular, as A^- is a subset of a sunflower, it is itself a sunflower. But there then must be some $k \in \omega$ such that $A^- \subseteq E_k$. However $\mathcal{E}_k \in F_{k,\ell} \subseteq D_{\ell+k}$ and so $\mathcal{E}_k$ contains no substructure isomorphic to $\mathcal{A}$ that is also a sunflower, getting us our contradiction. ☐

We can use Theorem 53 to show that several ages have the sunflower property.

Lemma 54. The following ages have the sunflower property:

(1) The collection of all finite graphs,
(2) The age $\mathcal{H}_J$ from Example 23.

Proof. (1) follows from Lemma 22 and Theorem 53, and (2) follows from Example 23 and Theorem 53. ☐

4.4 3-disjoint amalgamation property

Note that from Theorem 53, we can deduce general properties that will ensure an age has the sunflower property. In particular, we have the following.

Theorem 55. Suppose $\mathbf{K}$ is an age with only countably many isomorphism classes, that has unique unary quantifier-free types, and that has (HP) and (3-DAP). Then $\mathbf{K}$ has the sunflower property.

Proof. Because we are in a relational language, (3-DAP) implies (JEP). Therefore $\mathbf{K}$ has a Fraïssé limit $\mathcal{G}$. But then by Corollary 20, $\mathcal{G}$ is sunflowerable and so, by Theorem 53, $\mathbf{K}$ has the sunflower property. ☐

In the case of ages with (HP) and (3-DAP) we can also give concrete bounds on the size of a structure witnessing the sunflower property.

Theorem 56. Suppose

(1) $\mathbf{K}$ has (HP) and (3-DAP),
(2) for all $\mathcal{A} \in \mathbf{K}$ with $|\mathcal{A}| > 2$, $\mathbf{K}$ is indivisible at $\mathcal{A}$,
(3) β is a bound on indivisibility for $\mathbf{K}$, and
(4) for $\mathcal{A} \in \mathbf{K}$, $\gamma_{\mathcal{A}}$ is the function where $\gamma_{\mathcal{A}}(1) = |A|$ and $\gamma_{\mathcal{A}}(n+1) = \beta(\gamma_{\mathcal{A}}(n), (n+1)\cdot|A|)^{\leq |A|}$.

Then for all $\mathcal{A} \in \mathbf{K}$ and $n \in \omega$ there is a $\mathcal{B} \in \mathbf{K}$ with $|B| \leq \gamma_{\mathcal{A}}(n)$ such that $\mathcal{B}$ witnesses the n-sunflower property of $\mathbf{K}$ at $\mathcal{A}$.

Proof. We prove this by induction on n. First note that if $n = 1$, S is a set $\mathcal{A} \in \mathbf{K}$ and $f\colon A \to \mathcal{P}_1(S)$ is an injection, then $f[A]$ is a sunflower. Therefore $\mathcal{A}$ is a witness to this fact and $|A| \leq \gamma_{\mathcal{A}}(1) = |A|$.

We now assume the result holds for n in order to show it holds for $n+1$. Suppose $\mathcal{A} \in \mathbf{K}$. If $|A| = 2$ then, whenever S is a set and $f\colon \mathcal{A} \to \mathcal{P}_{n+1}(S)$, we have $f[A]$ is a sunflower. Therefore $\mathcal{A}$ witnesses that $\mathbf{K}$ has the $(n+1)$-sunflower property at $\mathcal{A}$ and $|A| = \gamma_{\mathcal{A}}(1) \leq \gamma_{\mathcal{A}}(n+1)$. Now suppose $|A| > 2$ and let $(a_i)_{i\in[\ell+1]}$ be an enumeration of $\mathcal{A}$. For $i \in [\ell]$ let $p_i = \mathrm{qtp}\big(a_{i+1}/(a_j)_{j\in[i]}\big)$. Let $\mathcal{B}^-$ witness that $\mathbf{K}$ has the n-sunflower property at $\mathcal{A}$ and $|B^-| \leq \gamma_{\mathcal{A}}(n)$. Let $\mathcal{B} \in \mathbf{K}$ be a structure witnessing the indivisibility of $\mathcal{B}^-$ in $\mathbf{K}$ of order $(n+1)\cdot|A|$. Further. suppose $|B| \leq \beta(|B^-|, (n+1)\cdot|A|) \leq \beta(\gamma_{\mathcal{A}}(n), (n+1)\cdot|A|)$.

We now define, for each $i \in [\ell+1]$, a structure $\mathcal{C}_i$ along with a collection V_i of valid sequences of length i of elements of $\mathcal{C}_i$ with the property that if $\boldsymbol{c} \in V_i$, then $\mathrm{qtp}(\boldsymbol{c}) = \mathrm{qtp}\big(\langle a_j\rangle_{j\in[i]}\big)$. Further, we will have $|V_i| \leq |B|^i$ and $|C_i| \leq |B| \cdot \sum_{j<i} |V_j|$. Let $\mathcal{C}_1 = \mathcal{A}{\restriction}_{\{a_0\}}$ and let $V_1 = \{\langle a_0\rangle\}$, i.e., $\langle a_0\rangle$ is the unique valid sequence of length 1. Suppose we have defined $\mathcal{C}_k$ and V_k for $k \in [\ell]$. For every $\boldsymbol{c} \in V_k$ we find a structures $\mathcal{B}_{\boldsymbol{c}}$, $\mathcal{D}_{\boldsymbol{c}}$ such that

(1) $B_{\boldsymbol{c}} \cap C_k = \varnothing$,
(2) $\mathcal{B}_{\boldsymbol{c}} \subseteq \mathcal{D}_{\boldsymbol{c}}$ and $\mathcal{C}_k \subseteq \mathcal{D}_{\boldsymbol{c}}$,

(3) $D_{\boldsymbol{c}} = B_{\boldsymbol{c}} \cup C_k$,
(4) $\mathcal{B}_{\boldsymbol{c}} \cong \mathcal{B}$, and
(5) for all $b \in \mathcal{B}_{\boldsymbol{c}}$ we have $\mathrm{qtp}(\boldsymbol{c}^\wedge b) = \mathrm{qtp}\big(\langle a_i\rangle_{i\in[k]}\big)$.

By Lemma 46, we have that $\mathbf{K}$ has unique unary quantifier-free types and so we can always find such a $\mathcal{D}_{\boldsymbol{c}}$ by Lemma 44. Further note we can assume, without loss of generality, for any two $\boldsymbol{c}_0, \boldsymbol{c}_1 \in V_k$ that $B_{\boldsymbol{c}_0} \cap B_{\boldsymbol{c}_1} = \varnothing$. Let $\mathcal{C}_{k+1}$ be a disjoint amalgamation of all of the $\mathcal{D}_{\boldsymbol{c}}$ for $\boldsymbol{c} \in V_k$. Note such a disjoint amalgamation exists in $\mathbf{K}$ by Lemma 41. We therefore have $|C_{k+1}| = |C_k| + |B| \cdot |V_k| \le |B| \cdot \sum_{i\in[k+1]} |V_i|$. Let $V_{k+1} = \{\boldsymbol{c}^\wedge b : \boldsymbol{c} \in V_k \text{ and } b \in \mathcal{B}_{\boldsymbol{c}}\}$. We then have $|V_{k+1}| \le |B| \cdot |V_k| \le |B|^{k+1}$. In particular, this implies $|C_\ell| \le |B| \cdot \sum_{i\in[\ell+1]} |B|^i \le \beta(\gamma_{\mathcal{A}}(n), (n+1)\cdot|A|)^{\le |A|}$. To finish the induction, it suffices to prove that $\mathcal{C}_\ell$ witnesses that $\mathbf{K}$ has the $(n+1)$-sunflower property at $\mathcal{A}$.

Suppose $\mathcal{C}^\circ_\ell$ is such that $\alpha\colon \mathcal{C}^\circ_\ell \cong \mathcal{C}_\ell$ and $C^\circ_\ell \subseteq \mathcal{P}_{n+1}(X)$ for some set X. Let $V^\circ_\ell = \{\alpha(\boldsymbol{c}) : \boldsymbol{c} \in V_\ell\}$. We now use V°_ℓ to attempt to construct a sunflower isomorphic to $\mathcal{A}$. For $x \in X$, let $E_x = \{y \in C^\circ_\ell : x \in y\}$. Call a sequence $\langle e_i\rangle_{i\in[k]}$ *separated* if for all $x \in X$ $|\{i : e_i \in E_x\}| \le 1$. We now have two cases.

Case 1: There is a $\langle c_i\rangle_{i\in[\ell+1]} \in V_\ell$ that is separated. In this case the map $a_i \mapsto c_i$ is an isomorphism between $\mathcal{A}$ and $\mathcal{C}^\circ_\ell\restriction_{\{c_i\}_{i\in[\ell+1]}}$. Further, because $\langle c_i\rangle_{i\in[\ell+1]}$ is separated, $c_i \cap c_j = \varnothing$ for $i < j \in [\ell+1]$. Therefore $\{c_i\}_{i\in[\ell+1]}$ is a sunflower.

Case 2: For all $\boldsymbol{c} \in V_\ell$, $\boldsymbol{c}$ is not separated. Let $P = \{\boldsymbol{c} \in \bigcup_{i\in[\ell+1]} V_i : \boldsymbol{c} \text{ is separated}\}$. Let $\boldsymbol{c} \le \boldsymbol{d}$ if $\boldsymbol{c}$ is an initial sequence of $\boldsymbol{d}$. $(P, \le)$ is then a finite partial ordering and so must have a maximal element $\boldsymbol{d}$. But by assumption, as $\boldsymbol{d}$ is separated, $\boldsymbol{d} \notin V_\ell$.

Let $\boldsymbol{d} = \langle d_i\rangle_{i\in[k]}$ where $k \in [\ell]$. Let $X_0 = \bigcup_{i\in[k]} d_i$, let $r = |X_0|$ and let $\gamma\colon X_0 \to [r]$ be a bijection. Note $r \le (n+1)\cdot|A|$. As $\boldsymbol{d}$ is maximal among $\bigcup_{i\in[\ell+1]} V_i$ that are separated, we have for all $b \in \mathcal{B}_{\boldsymbol{d}}$ that $\boldsymbol{d}^\wedge b$ is not separated. Therefore for all $b \in B_{\boldsymbol{d}}$, we have $b \cap X_0 \neq \varnothing$. Let $\zeta\colon B_{\boldsymbol{d}} \to [r]$ be any map such that $(\forall b \in B_{\boldsymbol{d}})\, \gamma^{-1}(\zeta(b)) \in b$. Because $\mathcal{B}$ is a witness to the indivisibility of $\mathcal{B}^-$ in $\mathbf{K}$ of order $(n+1)\cdot|A|$, there must be an $s \in [r]$ and a subset $B^-_{\boldsymbol{d}} \subseteq B_{\boldsymbol{d}} \cap \zeta^{-1}(s)$ with $B^-_{\boldsymbol{d}} \cong \mathcal{B}^-$. Let $z = \gamma^{-1}(s)$. Then for all $b \in B^-_{\boldsymbol{d}}$ we have $z \in b$. Let $G^-_{\boldsymbol{d}} = \{y \backslash \{z\} : y \in B^-_{\boldsymbol{d}}\}$. We then have the map $\nu_z\colon G^-_{\boldsymbol{d}} \to B^-_{\boldsymbol{d}}$ where $\nu_z(y) = y \cup \{z\}$ is an isomorphism. Let $\mathcal{G}^-_{\boldsymbol{d}}$ be the $\mathscr{L}$-structure with underlying set $G^-_{\boldsymbol{d}}$ where ν_z is an isomorphism of $\mathscr{L}$-structures.

Because $\mathcal{B}^-$ witnesses that $\mathbf{K}$ has the n-sunflower property at $\mathcal{A}$ and $\mathcal{G}^-_{\boldsymbol{d}} \cong \mathcal{B}^-$ there must be an $\mathcal{A}^* \subseteq \mathcal{G}^-_{\boldsymbol{d}}$ where $\mathcal{A}^* \cong \mathcal{A}$ and A^* is a sunflower. But then if we let $\mathcal{A}^\circ$ be $\nu_z[\mathcal{A}^*]$, we have $\mathcal{A}^\circ \subseteq \mathcal{C}^\circ_\ell$ and $\mathcal{A}^\circ \cong \mathcal{A}$. But we also have for all $a, b, c \in \mathcal{A}^*$, $\nu_z(a) \cap \nu_z(b) = (a \cap b) \cup \{z\} = (a \cap c) \cup \{z\} = \nu_z(a) \cap \nu_z(c)$. Therefore A° is also a sunflower.

Therefore $\mathcal{C}_\ell$ witnesses that $\mathbf{K}$ has the $(n+1)$-sunflower property at $\mathcal{A}$. ◻

5 Indivisibility

In this section, we discuss the relationship between indivisibility and being sunflowerable or being an age with a sunflower property. This will allow us to use our results to give new classes of structures that are indivisible. The following results follow from Proposition 10 and the corresponding facts about sunflowerable structures.

Lemma 57. Suppose

(1) $\mathcal{M}$ is an $\mathscr{L}$-structure with $|M|$ a regular cardinal,
(2) $\mathcal{M}$ is an $|M|$-saturated structure with quantifier-elimination,
(3) all elements of $\mathcal{M}$ have the same quantifier-free type, and
(4) $\mathcal{M}$ has (3-DAP).

Then $\mathcal{M}$ is indivisible.

Proof. This follows immediately from Corollary 20 and Proposition 10. ◻

We now show that being indivisible is a necessary condition for having the 2-sunflower property.

Proposition 58. If $\mathbf{K}$ has the 2-sunflower property, then $\mathbf{K}$ is indivisible at $\mathcal{A}$ whenever $\mathcal{A} \in \mathbf{K}$ with $|A| > 2$.

Proof. Suppose $\mathcal{A} \in \mathbf{K}$ and let $\mathcal{B}$ witness that $\mathbf{K}$ has the 2-sunflower property at $\mathcal{A}$. Note we can assume without loss of generality that $\bigcup B \cap [2] = \varnothing$. Suppose $\alpha \colon B \to [2]$. Let $\mathcal{B}' \in \mathbf{K}$ be the structure where $B' = \{\{b, \alpha(b)\} : b \in B\}$ and where the projection map $\pi_0 \colon B' \to B$ is an isomorphism. There must then be a subset $A' \subseteq B'$ with $\mathcal{B}\restriction_{A'} \cong \mathcal{A}$ and A' a sunflower. But as $|A| > 2$, all sunflowers in B' are either subsets of $\{\{b, 0\} : \alpha(b) = 0\}$ or subsets of $\{\{b, 1\} : \alpha(b) = 1\}$. Therefore we must have $|\alpha[A']| = 1$. The result then follows from the following claim: Suppose for all $\mathcal{A} \in \mathbf{K}$ there is a $\mathcal{B}_2 \in \mathbf{K}$ that witnesses the indivisibility of $\mathcal{A}$ in $\mathbf{K}$ of order 2, then $\mathbf{K}$ is indivisible.

We define $\mathcal{B}_n$, for $n > 2$, to be such that $\mathcal{B}_{n+1}$ witnesses the indivisibility of $\mathcal{B}_n$ in $\mathbf{K}$ of order 2. We now prove by induction on n that $\mathcal{B}_n$ witnesses the indivisibility of $\mathcal{A}$ in $\mathbf{K}$ of order n. First note that if $n = 2$, then this follows from our definition of $\mathcal{B}_2$.

Now suppose $\mathcal{B}_n$ witnesses the indivisibility of $\mathcal{A}$ in $\mathbf{K}$ of order n. Let $\alpha \colon B_{n+1} \to [n+1]$. Let $\gamma \colon [n+1] \to [2]$ be such that $\gamma(i) = 0$ for $i \in [n]$ and $\gamma(n) = 1$. By our definition of $\mathcal{B}_{n+1}$, there must be a $B_n^* \subseteq B_{n+1}$ such that $\mathcal{B}_n \cong \mathcal{B}_{n+1}\restriction_{B_n^*}$ with $|(\gamma \circ \alpha)[B_n^*]| = 1$. Suppose $B_n^* \subseteq (\gamma \circ \alpha)^{-1}(1)$, then $B_n^* \subseteq \alpha^{-1}[\{n\}]$. But, by construction, there must be an $A^* \subseteq B_n^*$ with $\mathcal{B}_n^*\restriction_{A^*} \cong \mathcal{A}$. Therefore $|\alpha[A^*]| = 1$ and we are done.

Now suppose $B_n^* \subseteq (\gamma \circ \alpha)^{-1}(0)$. Then $\alpha\restriction_{B_n^*} \colon B_n^* \to [n]$. Therefore by our inductive hypothesis there must be an $A^* \subseteq B_n^*$ such that $\mathcal{A} \cong \mathcal{B}_n^*\restriction_{A^*}$ and $|\alpha[A^*]| = 1$. □

Note in Proposition 58 that we cannot remove the assumption that $|A| > 2$, as for every set S every subset of $\mathcal{P}_n(S)$ of size 2 is a sunflower. As a consequence of Proposition 58 we also get the following, which was proved in [9] Proposition 2.23. Rehana Patel, using different methods, has independently also proved the following.

Lemma 59. Suppose $\mathbf{K}$ is an age with only countably many isomorphism classes, that has unique unary quantifier-free types, and that has (HP) and (3-DAP). Then $\mathbf{K}$ is indivisible.

Proof. This follows from Theorem 55 and Proposition 58. □

6 Conjectures

We end with a couple of conjectures. First recall, by Lemma 8, that the properties of being n-sunflowerable form a hierarchy. However, for all examples in this paper, this hierarchy collapses. This suggests the following conjecture.

Conjecture 60. If $\mathcal{M}$ is a 2-sunflowerable structure, then $\mathcal{M}$ is sunflowerable.

The definition of a sunflowerable structure suggests a higher dimensional analog.

Definition 61. Suppose $\mathcal{M}$ is an infinite $\mathscr{L}$-structure and either $k, n \in \omega$. We say $\mathcal{M}$ is *n-sunflowerable in dimension k* if, whenever S is a set and $f \colon M^k \to \mathcal{P}_n(S)$ is an injection, there is a subset $M_0 \subseteq M$ such that $\mathcal{M}\restriction_{M_0} \cong \mathcal{M}$ and $f[M_0^k]$ is a sunflower. We say $\mathcal{M}$ is *sunflowerable in dimension k* if it is n-sunflowerable in dimension k for every $n \in \omega$.

Conjecture 62. For every $n, k \in \omega$, a structure is n-sunflowerable in dimension k if and only if it is n-sunflowerable.

Theorem 36 characterizes those countable linear orderings that are sunflowerable. However, when we move to uncountable linear orderings we do not quite have such a characterization, as there are linear orderings that are $\mathbb{Q}_\kappa$-scattered but not κ-scattered. We therefore have the following conjecture.

Conjecture 63. If κ is regular, then all κ-dense linear orderings are sunflowerable.

For $n \geq 3$, recall $\mathcal{H}_n$ is the countable generic K_n-free graph. Then $\mathcal{H}_n$ is ultrahomogeneous and indivisible (see [17]). This then suggests the following conjecture.

Conjecture 64. For all $n \geq 3$, $\mathcal{H}_n$ is sunflowerable.

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