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The uncountability of the reals and the axiom of choice

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Abstract. The uncountability of the reals was first established by Cantor in what was later heralded as *the first paper on set theory*. Since the latter constitutes the de facto foundations of mathematics, the logical study of the uncountability of the reals is a worthy endeavour for historical, foundational, and conceptual reasons. In this paper, we shall study the following principle:

$\mathsf{NIN}_{[0,1]}$: there is no injection from the unit interval to the natural numbers.

We show that relatively strong logical systems cannot prove $\mathsf{NIN}_{[0,1]}$. In particular, the former system implies second-order arithmetic and fragments of the axiom of choice, including dependent choice. We also study the latter choice fragments in Kohlenbach's higher-order reverse mathematics.

1 Introduction

1.1 Aim and motivation

The fact that infinity comes in different sizes was established by Cantor in the first paper on set theory [5], in the form of the uncountability of $\mathbb{R}$. Given that set theory provides the current foundations of mathematics, the study of the uncountability of $\mathbb{R}$ is interesting for historical, foundational, and conceptual reasons. Below, we study the uncountability of $\mathbb{R}$ formulated as follows in Kohlenbach's *higher-order reverse mathematics* [13].

Principle 1 ($\mathsf{NIN}_{[0,1]}$)**.** For any $Y\colon [0,1]\to\mathbb{N}$, there are $x,y\in[0,1]$ such that $x\neq_{\mathbb{R}} y$ and $Y(x)=_{\mathbb{N}} Y(y)$.

The authors have shown in [20] that a relatively strong axiom system, namely $\mathsf{Z}_2^\omega+\mathsf{QF\text{-}AC}^{0,1}$, cannot prove $\mathsf{NIN}_{[0,1]}$. This system is introduced in Section 1.2 and implies second-order arithmetic and countable choice. The goal of this paper is to extend this negative result, namely to show that $\mathsf{NIN}_{[0,1]}$ is not provable in $\mathsf{Z}_2^\omega+\bigcup_\sigma \mathsf{QF\text{-}AC}^{\sigma,1}$ where the latter fragment of the axiom of choice is as follows.

Principle 2 ($\mathsf{QF\text{-}AC}^{\sigma,\tau}$)**.** For finite types σ,τ and quantifier-free φ:

$$(\forall f^\sigma)(\exists g^\tau)\,\varphi(f,g)\to(\exists \Phi^{\sigma\to\tau})(\forall f^\sigma)\,\varphi(f,\Phi(f)).$$

Here, $\mathsf{QF\text{-}AC}^{1,1}$ already gives rise to fragments of the axiom of dependent choice.

Finally, our negative result is established in Section 2. We also study $\mathsf{QF\text{-}AC}^{\sigma,1}$ and its kin in higher-order reverse mathematics in Section 3.

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1.2 Preliminaries

We introduce some required definitions for the below. We assume familiarity with Kohlenbach's higher-order reverse mathematics (abbreviated RM in the below), the base theory RCA_0^ω in particular. The original reference is [13], with a recent basic introduction in [27]. We note that real numbers are defined in RCA_0^ω in the same way as in second-order RM, i.e., as fast-converging Cauchy sequences. We stress that some of the below functionals were already studied by Hilbert & Bernays [10].

First of all, we consider the following axiom where the functional E is also called *Kleene's quantifier* $\exists^2$ and is discontinuous on Cantor space.

$$\big(\exists E\colon \mathbb{N}^\mathbb{N} \to \{0,1\}\big)\big(\forall f \in \mathbb{N}^\mathbb{N}\big)(E(f) = 0 \leftrightarrow (\exists n \in \mathbb{N})(f(n) = 0)). \tag{$\exists^2$}$$

We write $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\exists^2)$ and observe that the latter proves the same second-order sentences as ACA_0 (see [11]). We shall mostly work in ACA_0^ω, which is convenient, as a set of reals $X \subseteq \mathbb{R}$ is defined via $F_X\colon \mathbb{R} \to \{0,1\}$ where $x \in X \leftrightarrow F_X(x) = 1$ for all $x \in \mathbb{R}$. Over RCA_0^ω, $(\exists^2)$ is equivalent to (μ^2) (see [13]), where the later expresses

$$\big(\exists \mu\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}\big)\big(\forall f \in \mathbb{N}^\mathbb{N}\big)\big[(\exists n \in \mathbb{N})(f(n) = 0) \to f(\mu(f)) = 0\big]. \tag{μ^2}$$

Secondly, consider the following axiom:

$$\big(\exists \mathsf{S}\colon \mathbb{N}^\mathbb{N} \to \{0,1\}\big)\big(\forall f \in \mathbb{N}^\mathbb{N}\big)\big[\mathsf{S}(f) = 0 \leftrightarrow \big(\exists g \in \mathbb{N}^\mathbb{N}\big)(\forall n \in \mathbb{N})(f(\overline{g}n) = 0)\big]. \tag{S^2}$$

The functional S^2 is often called *the Suslin functional* (see [13; 1; 23]). By definition, the Suslin functional S^2 can decide whether a Σ^1_1-formula in normal form, i.e., as in the right-hand side of (S^2), is true or false. The system $\Pi^1_1\text{-}\mathsf{CA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\mathsf{S}^2)$ proves the same Π^1_3-sentences as $\Pi^1_1\text{-}\mathsf{CA}_0$ (see [23]).

Thirdly, we define the functional S^2_k which decides the truth or falsity of Σ^1_k-formulas in normal form; we also define the system $\Pi^1_k\text{-}\mathsf{CA}_0^\omega$ as $\mathsf{RCA}_0^\omega + (\mathsf{S}^2_k)$, where (S^2_k) expresses that S^2_k exists. We define Z_2^ω as $\bigcup_k \Pi^1_k\text{-}\mathsf{CA}_0^\omega$ as one possible higher-order version of Z_2. The functionals ν_n from [4, p. 129] are just S^2_n strengthened to return a witness (if existent) to the Σ^1_n-formula at hand. The operator ν_n is essentially Hilbert-Bernays' operator ν from [10, p. 479] restricted to Σ^1_n-formulas.

Fourthly, we introduce *Kleene's quantifier* $\exists^3$ as follows:

$$(\exists E)\big(\forall Y\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}\big)\big(E(Y) = 0 \leftrightarrow \big[\big(\exists f \in \mathbb{N}^\mathbb{N}\big)(Y(f) = 0)\big]\big). \tag{$\exists^3$}$$

Both $\mathsf{Z}_2^\Omega \equiv \mathsf{RCA}_0^\omega + (\exists^3)$ and $\mathsf{Z}_2^\omega \equiv \bigcup_k \Pi^1_k\text{-}\mathsf{CA}_0^\omega$ are conservative over Z_2 (see [11]). The functional E from $(\exists^3)$ is also called "$\exists^3$", and we use the same convention for other functionals. Hilbert–Bernays' operator ν (see [10, p. 479]) is essentially Kleene's $\exists^3$, modulo a non-trivial fragment of the axiom of (quantifier-free) choice.

2 A more negative result

We establish the negative result regarding the uncountability of $\mathbb{R}$ sketched in Section 1.1. In particular, we show that $\mathsf{Z}_2^\omega + \bigcup_\sigma \mathsf{QF\text{-}AC}^{\sigma,1}$ does not prove $\mathsf{NIN}_{[0,1]}$. To this end, we recall the model $\mathbf{P}$ introduced in [20] in Section 2.1. We show in Section 2.2 that $\mathbf{P}$ satisfies the aforementioned instances of the axiom of choice.

2.1 A model of finite type arithmetic

We introduce the model $\mathbf{P}$ from [20] as it will be essential to our main result in Section 2.2. We first make our notion of "computability" precise as follows.

(I) We adopt ZFC, i.e., Zermelo–Fraenkel set theory with the axiom of choice, as the official metatheory for all results, unless explicitly stated otherwise;

(II) We adopt Kleene's notion of *higher-order computation* as given by his nine clauses S1–S9 (see [14, Chapter 5] or [12]) as our official notion of computable.

For those familiar with *Turing* computability, Kleene's S1–S9 in a nutshell are as follows: the schemes S1–S8 merely introduce (higher-order) primitive recursion, while S9 essentially states that the recursion theorem holds.

We refer to [14] for a thorough and recent overview of higher-order computability theory. We do sketch one of our main techniques, called *Gandy selection*. Intuitively speaking, this method expresses that S1–S9 computability satisfies an effective version of the axiom of choice.

Secondly, we have the following definition of "model".

Definition 3. A *type structure* Tp is a sequence $(\mathsf{Tp}[k])_{k\in\mathbb{N}}$ as follows.

(i) $\mathsf{Tp}[0] = \mathbb{N}$;
(ii) For all $k \in \mathbb{N}$, $\mathsf{Tp}[k+1]$ is a set of functions $\Phi\colon \mathsf{Tp}[k] \to \mathbb{N}$.

We note that Tp involves only total objects. The Kleene schemes can be interpreted for all type structures, by simply relativizing the definition. One of our applications of type structures is that they serve as models for fragments of higher-order arithmetic, structures over the *language of finite types*. While the Kleene schemes are defined for pure types, the language of finite types is richer. However, assuming some modest closure properties of a type structure Tp, the extension to the finite types is unique (see [14, § 4.2]). This is the case when Tp is *Kleene closed* as in Definition 4 right below.

Definition 4 (Kleene computability)**.**

(i) Let Tp be a type structure, let $\varphi\colon \mathsf{Tp}[k] \to \mathbb{N}$, and let $\vec{\Phi}$ be in Tp. We say that φ is *Kleene computable* in $\vec{\Phi}$ (over Tp) if there is an index e such that for all $\xi \in \mathsf{Tp}[k]$ we have that $\{e\}(\xi, \vec{\Phi}) = \varphi(\xi)$;
(ii) The type structure Tp is *Kleene closed* if for all k and all $\varphi\colon \mathsf{Tp}[k] \to \mathbb{N}$ that are Kleene computable in elements in Tp, we have that $\varphi \in \mathsf{Tp}[k+1]$.

When a type structure Tp is Kleene closed, it will have a canonical extension to an interpretation $\mathsf{Tp}[\sigma]$ for all finite types σ. This is folklore and is discussed at length in [14, § 4.2]. We use Tp^* to denote this unique extension. What is important to us is that if Tp is Kleene closed, then Tp^* is a model of RCA_0^ω and all terms in Gödel's T have canonical interpretations in Tp^*.

Thirdly, the following (folklore) theorem shows that we have a high degree of flexibility when defining type structures from sets of functionals.

Theorem 5. Let $A \subseteq \mathbb{N}^\mathbb{N}$ and let B be a set of functionals $F\colon A \to \mathbb{N}$. Assume that all f computable in a sequence from B and A are in A. Then there is a Kleene closed type structure Tp such that $A = \mathsf{Tp}[1]$ and $B \subseteq \mathsf{Tp}[2]$.

Fourthly, we introduce a version of *Gandy selection*, first proved in [8]. Intuitively, $\lambda G.\{d\}(F, G)$ as in Theorem 6 is a (partial) choice function with the biggest possible domain. A functional F^2 is *normal* if it computes $\exists^2$.

Theorem 6 (Gandy Selection)**.** Let F^2 be normal. Let $A \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}^\mathbb{N}}$ and e be such that $(a, G^2) \in A$ if and only if $\{e\}(F, G, a)$ terminates (A is semi-computable in F). Then there is an index d such that $\{d\}(F, G)$ terminates if and only if there exists $a \in \mathbb{N}$ such that $(a, G) \in A$, and then $\{d\}(F, G)$ is one of these numbers.

Proof. See [14, § 5.4] or [22, p. 245]. □

Remark 7. We need this strong version of Gandy selection in the proof of Lemma 13. When F is normal, computing relative to F and G^2 satisfies *stage comparison*, a soft requirement for Gandy selection. Replacing G with an arbitrary functional of type ≥ 3, stage comparison will not be available anymore, and thus neither Gandy selection. We can of course replace $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$ by $\mathbb{N}^\mathbb{N}$ in Theorem 6, as we do in the proof of Corollary 8.

Several of our applications of Gandy selection are based on the following corollary. Intuitively speaking, the functional G computes an F-index for f.

Corollary 8. Let F^2 be normal. Then there is a partial functional G computable in F which terminates if and only if the input $f \in \mathbb{N}^{\mathbb{N}}$ is computable in F, and such that we have $G(f) = e \to (\forall a \in \mathbb{N})(f(a) = \{e\}(F, a))$.

Proof. When F is normal, the relation $(\forall a \in \mathbb{N})(f(a) = \{e\}(F, a))$ is clearly semi-computable, and we can apply Gandy Selection. □

Note that the functional G is always injective. Of course, these results are equally valid for all Kleene closed type structures, and we may replace $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ in Theorem 6 by any finite product of pure types ≤ 2.

Next, we define the Kleene closed type structure $\mathbf{P}$, which is crucial for the below independence results involving Z_2^ω. We note that $\mathbf{P}$ is constructed under the set-theoretical assumption that $\mathsf{V} = \mathsf{L}$. There is no harm in this, since what is of interest is the logic of the structure, which statements are true and which are false, and our results will not depend on the assumption that $\mathsf{V} = \mathsf{L}$; they are proved within ZF. Now, recall the functionals S_k^2 from Section 1.2. We use the assumption $\mathsf{V} = \mathsf{L}$ motivated by the following fact from set theory.

Lemma 9 ($\mathsf{V} = \mathsf{L}$)**.** Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be closed under computability relative to all S_k^2. Then all Π_n^1-formulas are absolute for A for all n.

Proof. For $n \leq 2$, this is a general fact independent of the assumption $\mathsf{V} = \mathsf{L}$, and for $n > 2$ it is a consequence of the existence of a Δ_2^1-well-ordering of $\mathbb{N}^{\mathbb{N}}$. □

Definition 10 ($\mathsf{V} = \mathsf{L}$)**.** Let S_ω^2 be the join of all S_k^2, and let $\mathbf{P}$ be the Kleene closed type-structure, as obtained from Theorem 5, where $\mathbf{P}[1]$ is the set of functions computable in S_ω^2 and the restriction of S_ω^2 to $\mathbf{P}[1]$ is in $\mathbf{P}[2]$.

The model $\mathbf{P}$, under another name, has been used to prove [18, Theorem 4.3]. Recall the unique extension Tp^* of Tp introduced below Definition 4. The principle $\mathsf{NIN}_{2^{\mathbb{N}}}$ expresses that there is no injection from Cantor space to the naturals.

Theorem 11. The type structure $\mathbf{P}^*$ derived from $\mathbf{P}$ as defined above is a model for $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1} + \neg\mathsf{NIN}_{2^{\mathbb{N}}}$.

Proof. We assume that $\mathsf{V} = \mathsf{L}$, which implies that all Π_n^1-formulas are absolute for $\mathbf{P}[1]$. Since $\mathbf{P}[1]$ is closed under computability relative to each S_k^2, we have that $\mathbf{P}[1]$ satisfies all Π_n^1-comprehension axioms. Now assume that $(\forall n^0)(\exists f^1)Q\big(n, f, \vec{\Phi}\big)$ is true in $\mathbf{P}$, where Q is quantifier-free and $\vec{\Phi}$ is a list of parameters from $\mathbf{P}$. Since all functionals in $\vec{\Phi}$ are computable in S_ω^2, the set

$$S \coloneqq \big\{(n, f) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} : Q\big(n, f, \vec{\Phi}\big)\big\}$$

is computable in S_ω^2. Here, we use the substitution theorem for S1–S9 from [12] relativised to the model $\mathbf{P}$. Furthermore, the set TOT of indices e^0 for total functions $f \in \mathbb{N}^{\mathbb{N}}$ relative to any total functional F^2 is semi-computable in F since we have

$$e \in \mathsf{TOT} \leftrightarrow \big[F\big(\lambda a^0.\{e\}(a, F)\big) \text{ terminates}\big].$$

As a consequence, and technically by a second use of the aforementioned substitution theorem, the following set is semi-computable in S_ω^2:

$$R \coloneqq \big\{(n, e) \in \mathbb{N}^2 : \big(\exists f \in \mathbb{N}^{\mathbb{N}}\big)\big[Q\big(n, f, \vec{\Phi}\big) \wedge (\forall a \in \mathbb{N})(f(a) = \{e\}(\mathsf{S}_\omega^2, a))\big]\big\}.$$

Moreover, we have that $(\forall n^0)(\exists e^0)[(n, e) \in R]$. By assumption and Gandy selection, there is a function g computable in S_ω^2 such that $R(n, g(n))$ for all n^0. If $G(n)$ is the function f computed from S_ω^2 with index $g(n)$, we have that $G^{0\to 1} \in \mathbf{P}$ witnesses this instance of quantifier-free choice.

Regarding $\mathsf{NIN}_{2^{\mathbb{N}}}$, recall that $\mathbf{P}$ consists of all objects computable in S^2_ω, the union of all S^2_k. Thus, for any $f \in 2^{\mathbb{N}} \cap \mathbf{P}$ there is some (unique minimal) $e_f \in \mathbb{N}$ such that the e_f-th Kleene algorithm computes f in terms of S^2_ω; by Gandy selection, the choice function $\lambda f^1.e_f$ is already part of $\mathbf{P}$ and provides an injection from $2^{\mathbb{N}} \cap \mathbf{P}$ to $\mathbb{N}$, i.e., $\mathsf{NIN}_{2^{\mathbb{N}}}$ is false in $\mathbf{P}$. More formally, $\mathbf{P}[2]$ contains a functional $G\colon A \to \mathbb{N}$ that is *injective*, which is a direct consequence of Corollary 8, using S^2_ω for F. □

Finally, it is straightforward to prove the equivalence $\mathsf{NIN}_{[0,1]} \leftrightarrow \mathsf{NIN}_{2^{\mathbb{N}}}$, say over ACA^ω_0 (see [25] for a proof). In particular, the latter system provides a uniform mechanism for converting reals in $[0,1]$ to binary representation.

2.2 Main result

We show that $\mathsf{NIN}_{[0,1]}$ cannot be proved in $\mathsf{Z}^\omega_2 + \bigcup_\sigma \mathsf{QF\text{-}AC}^{\sigma,1}$ by showing that the model $\mathbf{P}$ from Section 2.1 satisfies the latter but not the former.

First of all, we establish two lemmas needed for our main result. Recalling that the only predicate of our language is equality at base type 0, the following claim is just an observation that does not require a proof.

Lemma 12. Let Δ be a quantifier-free formula with higher-order parameters $\vec{\Phi}$ and free variables x^τ and y^1. Then there is a functional $\Psi^{\tau\to 2}$, computable in $\vec{\Phi}$, such that for each φ^τ and f^1 we have

$$\Delta\big(\varphi, f, \vec{\Phi}\big) \leftrightarrow \Psi(\varphi)(f) = 0.$$

Clearly, Lemma 12 is also valid in $\mathbf{P}$, and the same is true for the next one.

Lemma 13. Let F be a normal functional of type 2. Then there is a partial functional Θ of type $2 \to 1$ such that for all G of type 2, the following are equivalent:

(a) the function $\Theta(G)$ is total and $G(\Theta(G)) = 0$;
(b) there exists a f^1 computable in F and G such that $G(f) = 0$.

Proof. This is proved via Gandy selection. For G of type 2, define

$$A_G \coloneqq \big\{e \in \mathbb{N} : \lambda a^0.\{e\}(a, F, G) \text{ is total and } G \text{ maps the former to } 0\big\}.$$

Clearly, A_G is semi-computable in F and G, uniformly in G. By Gandy selection, there is a partial computable functional ξ selecting $e_0 \in A_G$ whenever the latter is nonempty. Composing ξ with $e \mapsto \lambda a.\{e\}(a, F, G)$ yields the required Θ. □

We may now prove the main theorem of this section.

Theorem 14. The system $\mathsf{Z}^\omega_2 + \bigcup_\sigma \mathsf{QF\text{-}AC}^{\sigma,1}$ does not prove $\mathsf{NIN}_{[0,1]}$.

Proof. We let $\mathbf{P}$ be the model from Definition 10. By Theorem 11, $\mathbf{P}$ satisfies $\mathsf{Z}^\omega_2 + \neg\mathsf{NIN}_{[0,1]}$. Now assume that $\mathbf{P} \vDash (\forall x^\tau)(\exists y^1)\,\Delta(x, y, \vec{\Phi})$, where Δ is quantifier-free. We work inside $\mathbf{P}$ and use the above lemmas. Consider Ψ provided by applying Lemma 12 for Δ, then Ψ is computable in S^2_ω, since all parameters in Δ are computable in S^2_ω. Similarly, apply Lemma 13 for $F = \mathsf{S}^2_\omega$ and let Θ be the resulting functional. By assumption on Δ, we have $(\forall x^\tau)\,\Delta\big(x, \Theta(\Psi(x)), \vec{\Phi}\big)$. Moreover, $\lambda x^\tau.\Theta(\Psi(x))$ is computable in S^2_ω, and therefore already included in $\mathbf{P}$. □

Remark 15. We cannot replace $\mathsf{QF\text{-}AC}^{\sigma,1}$ by $\mathsf{QF\text{-}AC}^{\sigma,\tau}$ for types τ of any higher-order in the above argument for two reasons. One reason is discussed in Remark 7. The other reason is that a normal functional of type 2, like S^2_ω, can decide equality over $\mathbb{N}^\mathbb{N}$, but not for types at higher levels.

The theorem has some interesting implications for the following fragment of the axiom of dependent choice.

Principle 16 ($\mathsf{QF\text{-}DC}^{1,1}$)**.** Let φ be quantifier-free and such that $(\forall n \in \mathbb{N})\big(\forall f \in \mathbb{N}^{\mathbb{N}}\big)\big(\exists g \in \mathbb{N}^{\mathbb{N}}\big)\,\varphi(n,f,g)$. Then there is $(f_n)_{n\in\mathbb{N}}$ such that $(\forall n \in \mathbb{N})\,\varphi(n,f_n,f_{n+1})$.

The axiom $\mathsf{QF\text{-}DC}^{1,1}$ is similar in kind to Kohlenbach's fragment of countable choice $\mathsf{QF\text{-}AC}^{0,1}$ from [13]. We now have the following immediate corollary.

Corollary 17. The system $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}DC}^{1,1}$ cannot prove $\mathsf{NIN}_{[0,1]}$.

Proof. We shall show that $\mathsf{QF\text{-}DC}^{1,1}$ follows from $\mathsf{QF\text{-}AC}^{1,1}$ and $(\mathbf{R}_1)$, where the latter formalises primitive recursion of type $1 \to 1$ objects. The recursor constant $\mathbf{R}_\sigma$ is defined as follows in general:

$$\mathbf{R}_\sigma(f,g,0) \coloneqq g \text{ and } \mathbf{R}_\sigma(f,g,n+1) \coloneqq f(n,\mathbf{R}_\sigma(f,g,n)), \tag{$\mathbf{R}_\sigma$}$$

where g^σ and $f^{(0\times\sigma)\to\sigma}$ are arbitrary and σ is any finite type. Since Gödel's T is included in S1–S9 computability, the corollary follows from (the proof of) the theorem. For fixed quantifier-free φ, apply $\mathsf{QF\text{-}AC}^{1,1}$ to $(\forall n \in \mathbb{N})\big(\forall f \in \mathbb{N}^{\mathbb{N}}\big)\big(\exists g \in \mathbb{N}^{\mathbb{N}}\big)\,\varphi(n,f,g)$. Let $\lambda n.\lambda f.\Phi(n,f)$ be the resulting functional and use $\mathbf{R}_1$ to define $f_0 \coloneqq 00\ldots$ and $f_{n+1} \coloneqq \Phi(n,f_n)$, as required for $\mathsf{QF\text{-}DC}^{1,1}$. □

In conclusion, Theorem 14 suggests that we may use $\bigcup_\sigma \mathsf{QF\text{-}AC}^{\sigma,1}$ and $\mathsf{QF\text{-}DC}^{1,1}$ in the RM of $\mathsf{NIN}_{[0,1]}$ and beyond, which is the content of the following section.

3 Some applications

We discuss how quantifier-free choice as in $\mathsf{QF\text{-}AC}^{\sigma,1}$ is useful in higher-order RM. For instance, showing that a set is uncountable can be done in various ways and converting between these seems to require fragments of $\mathsf{QF\text{-}AC}^{\sigma,1}$.

In more detail, we establish new equivalences for $\mathsf{NIN}_{[0,1]}$ using $\mathsf{QF\text{-}AC}^{\sigma,1}$ in Section 3.1 and provide an overview of how $\mathsf{QF\text{-}AC}^{\sigma,1}$ has been (or can be) used in RM in Section 3.2. We stress that the negative results in Section 2.2 justify the use of $\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{\sigma,1}$ as a base theory for the RM of $\mathsf{NIN}_{[0,1]}$ and stronger principles. We also obtain an equivalence for $\mathsf{QF\text{-}DC}^{1,1}$ in Theorem 21.

3.1 Quantifier-free choice in the base theory

We establish some equivalences for $\mathsf{NIN}_{[0,1]}$ using $\mathsf{QF\text{-}AC}^{\sigma,1}$ in the base theory. We shall consider metric and second-countable spaces, which have been previously studied in RM (see [30; 7; 29; 28; 27]). The associated definitions in higher-order RM are the textbook ones (see [29; 28; 27]), i.e., we shall not introduce the former as they are well-known.

First of all, the following result seems to crucially depend on $\mathsf{QF\text{-}AC}^{1,1}$; Theorem 18 (c) is well-known from textbooks (see [15, Chapter 3, § 27, Exercise 4; 21, Chapter 2, Exercise 19]).

Theorem 18 ($\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{1,1}$)**.** The following are equivalent.

(a) The uncountability of the reals as in $\mathsf{NIN}_{[0,1]}$.
(b) For $X \subseteq \mathbb{R}$, if $f\colon X \to [0,1]$ is surjective, then X is uncountable.[1]
(c) Let (M,d) be a metric space with $M \subseteq \mathbb{R}$ and let $A \subseteq M$ be connected and satisfy $|A| \geq 2$. Then A is uncountable.[1]
(d) Let (M,d) be a connected metric space with $M \subseteq \mathbb{R}$ and $f\colon M \to \mathbb{R}$ continuous. If $f(M)$ exists and has two distinct elements, it is uncountable.

[1] Similar to $\mathsf{NIN}_{[0,1]}$, "A is uncountable" for $A \subseteq M$ in a metric space (M,d) means that there is no injection from A to $\mathbb{N}$, i.e., for any $Y\colon M \to \mathbb{N}$, there are $x,y \in A$ with $x \neq_M y \wedge Y(x) = Y(y)$. Note that $d(x,y) =_{\mathbb{R}} 0 \leftrightarrow x =_M y$ by the definition of metric space.

(e) The previous item for $f\colon M \to M_0$ where (M_0, d_0) is any metric space.
(f) Let (M_i, d_i) for $i = 0, 1$ be metric spaces with $M_i \subseteq \mathbb{R}$, $|M_0| \geq 2$, and M_0 connected, and let $f\colon M_1 \to M_0$ be surjective. Then M_1 is uncountable.
(g) Any connected functionally Hausdorff second-countable space over the reals with at least two points is uncountable.

We do not need $\mathsf{QF\text{-}AC}^{1,1}$ to prove (a) $\to$ (d).

Proof. That items (b)–(g) imply (a) is straightforward.

(a) $\to$ (b). Let $X \subseteq \mathbb{R}$ and $f\colon X \to [0,1]$ be as in the latter and apply $\mathsf{QF\text{-}AC}^{1,1}$ to $(\forall r \in [0,1])(\exists x \in X)(f(x) =_{\mathbb{R}} r)$ to obtain $\Phi^{1\to 1}$. In case $Y\colon \mathbb{R} \to \mathbb{N}$ is injective on X, then $\lambda r.Y(f(r))$ is injective on $[0,1]$, as required.

(a) $\to$ (c). Suppose the latter is false, i.e., let $A \subseteq M$ be countable and connected. Let $a, b \in A$ be distinct points and consider the following

$$(\exists \varepsilon \in (0, d(a,b)))(\forall x \in A)(d(a,x) \neq_{\mathbb{R}} \varepsilon). \tag{3.1}$$

Let ε be as in (3.1); and consider the open sets $U \coloneqq \{x \in A : d(a,x) < \varepsilon\}$ and $V \coloneqq \{x \in A : d(a,x) > \varepsilon\}$. Since $U \cup V = A$ and $U, V \neq \varnothing$, this contradicts the connectedness of A, i.e., (3.1) must be false. Now apply $\mathsf{QF\text{-}AC}^{1,1}$ to the negation of (3.1), as follows:

$$(\forall \varepsilon \in (0, d(a,b)))(\exists x \in A)(d(a,x) =_{\mathbb{R}} \varepsilon), \tag{3.2}$$

and let $\Phi^{1\to 1}$ be the resulting choice function. Since A is countable, there is $Y\colon M \to \mathbb{N}$ that is injective on A. Then $\lambda x.Y(\Phi(x))$ is an injection from the real interval $(0, d(a,b))$ to $\mathbb{N}$, which contradicts $\mathsf{NIN}_{[0,1]}$.

(a) $\to$ (d). We prove that for continuous $f\colon M \to \mathbb{R}$, $f(M)$ is connected if M is. Indeed, let M be connected and suppose $f(M) = U \cup V$ where $U, V \subseteq \mathbb{R}$ are open and non-empty. Now consider $X = f^{-1}(U)$ and $Y = f^{-1}(V)$, which are open, as topological continuity and epsilon-delta continuity are equivalent by definition in RCA_0^ω. Since M is connected, $M = X \cup Y$ implies that there is $z \in X \cap Y$, yielding that $f(z) \in U \cap V$, as required for the connectedness of $f(M)$. If $|f(M)| > 2$, then there are $x_0, y_0 \in M$ such that $f(x_0) \neq_{\mathbb{R}} f(y_0)$. To show that $(f(x_0), f(y_0)) \subseteq f(M)$, suppose $w \in (f(x_0), f(y_0))$ and $w \notin f(M)$. Then $U = \{x \in M : f(x) >_{\mathbb{R}} w\}$ and $V = \{x \in M : f(x) <_{\mathbb{R}} w\}$ are open, since f is continuous, contradicting that M is connected. Hence, $f(M)$ contains an interval and is therefore uncountable by $\mathsf{NIN}_{[0,1]}$, as required.

(a) $\to$ (e). Use the same proof as above to show that $f(M)$ is connected and apply (c).

(a) $\to$ (f). Let $f\colon M_1 \to M_0$ be as in the latter, i.e., $(\forall y \in M_0)(\exists x \in M_1)(f(x) = y)$; applying $\mathsf{QF\text{-}AC}^{1,1}$ yields $\Phi^{1\to 1}$ such that $y = \Phi(x)$ in the latter. In case (M_1, d) is countable, there is $Y\colon \mathbb{R} \to \mathbb{N}$ such that $Y(x) = Y(y)$ implies $x =_{M_1} y$ for any $x, y \in M_1$. Now define $Z\colon \mathbb{R} \to \mathbb{N}$ as $\lambda x.Y(\Phi(x))$, which satisfies $Z(x) = Z(y) \to x =_{M_0} y$ for all $x, y \in M_0$, essentially by definition. Hence (M_0, d) is countable, contradicting (c), as required.

(a) $\to$ (g). Let X be functionally Hausdorff, i.e., for any two distinct points $x, y \in X$ there is continuous $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. Now if $x_0 \in (0,1)$ is not in the range of f, then the open sets $U = f^{-1}([0, x_0))$ and $V = f^{-1}((x_0, 1])$ are non-empty, disjoint, and such that $X = U \cup V$, a contradiction. Now apply $\mathsf{QF\text{-}AC}^{1,1}$ to $(\forall r \in [0,1])(\exists x \in X)(f(x) =_{\mathbb{R}} r)$ to obtain $\Phi^{1\to 1}$ such that $x = \Phi(r)$ in the previous formula. Now, if $Y\colon X \to \mathbb{N}$ is injective, then so is $Z\colon [0,1] \to \mathbb{N}$, defined by $Z(r) \coloneqq Y(\Phi(r))$, contradicting $\mathsf{NIN}_{[0,1]}$. □

We can replace the existential quantifier in (3.2) by "$(\exists! x \in A)$", where uniqueness is, however, expressed relative to $=_M$. We also obtain weaker results as follows.

Corollary 19 (ACA_0^ω)**.** Items (b)–(g) can be proved for "uncountable" replaced by "non-enumerable".

Proof. We first consider (b) formulated with "X is non-enumerable". Now if X were enumerable, say given by the sequence $(x_n)_{n\in\mathbb{N}}$, the surjectivity of $f\colon X \to \mathbb{R}$ implies $(\forall r \in [0,1])(\exists n \in \mathbb{N})(f(x_n) =_{\mathbb{R}} r)$. Hence, $(f(x_n))_{n\in\mathbb{N}}$ enumerates $[0,1]$, which even RCA_0 disproves (see [30, Theorem II.4.9]). A similar proof works for (c): if A is given by a sequence $(a_n)_{n\in\mathbb{N}}$, then (3.2) implies $(\forall \varepsilon \in (0, d(a,b)))(\exists n \in \mathbb{N})(d(a,a_n) =_{\mathbb{R}} \varepsilon)$. Hence, the interval $(0, d(a,b))$ can be enumerated via $(d(a,a_n))_{n\in\mathbb{N}}$, a contradiction as for (b). The other items are proved in the same way. □

Next, we have the following theorem where we recall that $(\mathbf{R}_1)$ formalises primitive recursion of type $1 \to 1$ objects (see Corollary 17). Theorem 20 (c) is studied in textbooks (see [15, Theorem 27.7]) and second-order RM via codes (see [30, Corollary II.5.9]).

Theorem 20 $(\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{1,1} + (\mathbf{R}_1))$**.** The following are equivalent.

(a) The uncountability of the reals as in $\mathsf{NIN}_{[0,1]}$.
(b) A perfect set of reals is not countable.
(c) A complete metric space (M,d) with $M \subseteq \mathbb{R}$ and with *no isolated points*, is uncountable.[1]
(d) The previous item restricted to *separable* spaces.
(e) Let (M_i, d_i) be complete metric spaces without isolated points and $M_i \subseteq \mathbb{R}$. For continuous $f\colon M_0 \to M_1$, if the set $f(M_0)$ exists, it is uncountable.
(f) The previous item restricted to sequential continuity.

We do not need $(\mathbf{R}_1)$ to establish (a) → (b) or (a) → (d).

Proof. That (a) follows from the other items is straightforward.

(a) → (b) & (c). To this end, we first observe two applications of $\mathsf{QF\text{-}AC}^{1,1}$ for a metric space (M,d) as in (c):

(i) (M,d) has no isolated points, i.e., $(\forall x \in M, k \in \mathbb{N})(\exists y \in M)\big(0 < d(x,y) < \frac{1}{2^k}\big)$. Hence, there is $\Phi_0^{1\to 1}$ such that $y =_1 \Phi_0(x,k)$ in the latter.
(ii) (M,d) is complete, i.e., we have

$$(\forall (x_n)_{n\in\mathbb{N}})(\exists x \in M)[(x_n)_{n\in\mathbb{N}} \text{ is Cauchy} \to \lim_{n\to\infty} x_n =_M x]. \tag{3.3}$$

The formula in square brackets in (3.3) is arithmetical, i.e., there is $\Phi_1^{1\to 1}$ such that $x =_1 \Phi_1(\lambda n.x_n)$ in (3.3).

One standard technique to show that M is uncountable is then as follows: one constructs an injection $f\colon 2^{\mathbb{N}} \to M$ based on the below items.

(i) Use Φ_0 and $\mathbf{R}_1$ to define a sequence of pairwise different points $(x_\sigma)_{\sigma\in 2^{<\mathbb{N}}}$ in M such that $x_{\sigma*i*j} \in B\big(x_{\sigma*i}, \frac{1}{2}d(x_{\sigma*0}, x_{\sigma*1})\big)$.
(ii) For $\alpha \in 2^{\mathbb{N}}$, define $f(\alpha) = x_\alpha$ in case α has a tail of zeros. Otherwise, use Φ_1 to define $f(\alpha)$ as the limit of the sequence $(x_{\overline{\alpha}n})_{n\in\mathbb{N}}$ in M.
(iii) Verify that $f\colon 2^{\mathbb{N}} \to M$ is an injection by definition.

Since the space $2^{\mathbb{N}}$ does not really involve representations (in contrast to $\mathbb{R}$), the issue of extensionality relative to $=_M$ does not pose problems.

(a) → (d). For (d) (and the final sentence of the theorem), observe that the sequence $(x_\sigma)_{\sigma\in 2^{<\mathbb{N}}}$ is readily defined in terms of a dense subsequence of M, say over ACA_0^ω.

(a) → (e). Let (M_i, d_i) and $f\colon M_0 \to M_1$ be as in (e). By the continuity of f and the completeness of M_0, $f(M_0)$ is complete and cannot have isolated points. One seems to need $\mathsf{QF\text{-}AC}^{0,1}$ for the latter completeness. Hence, (c) implies (e), using $\mathsf{QF\text{-}AC}^{1,1}$.

(a) → (f). The proof using sequential continuity is essentially the same as above, again using $\mathsf{QF\text{-}AC}^{0,1}$ in an essential way. □

Next, we formulate an equivalence for $\mathsf{QF\text{-}DC}^{1,1}$ based on [3; 9], where we focus on what is exactly needed for equivalences to $\mathsf{QF\text{-}DC}^{1,1}$. It should be possible to obtain more equivalences based on the results in [26; 17]. The notion of *R2-open set* in Theorem 21 (b) was first introduced in [17].[2]

Theorem 21 (ACA_0^ω)**.** The following are equivalent.

(a) The principle $\mathsf{QF\text{-}DC}^{1,1}$.
(b) The Baire category theorem for metric spaces (M,d) with $M \subseteq \mathbb{N}^\mathbb{N}$ and sequences of dense and R2-open sets.
(c) (b) for M and the graph of d defined by arithmetical formulas.

Proof. (a) $\to$ (b). The standard (constructive) proof of the Baire category theorem is well-known (see [3, p. 87]). Modulo coding, this is the proof used in second-order RM (see [30, Theorem II.5.8]). The same proof goes through for complete metric spaces as in (b) if we use $\mathsf{QF\text{-}DC}^{1,1}$ to define the Cauchy sequence in the former proof. Thus, (b) follows from $\mathsf{QF\text{-}DC}^{1,1}$.

(b) $\to$ (a). Let M be the set of all sequences in Baire space (readily coded as a subset of Baire space), i.e., M consists of all objects of type $0 \to 1$. The metric $d\colon M^2 \to \mathbb{R}$ is then as follows for any $f, g \in M$:

$$d(f,g) = \begin{cases} 0 & \text{if } (\forall n \in \mathbb{N})(f(n) =_1 g(n)), \\ \dfrac{1}{2^{(\mu n)(f(n) \neq_1 g(n))}} & \text{otherwise.} \end{cases}$$

That (M,d) is complete follows by noting that a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in M converges to $g \in M$ defined as $g(n) \coloneqq f_{F(n)}(n)$ where F^1 is such that

$$(\forall k \in \mathbb{N})(\forall m,n \geq F(k))\big(d(f_n,f_m) < \tfrac{1}{2^k}\big).$$

Now let φ be quantifier-free and such that $\big(\forall x \in \mathbb{N}^\mathbb{N}\big)\big(\exists y \in \mathbb{N}^\mathbb{N}\big)\,\varphi(x,y)$ and define sets $O_n \subseteq M$, for any $f^{0\to 1} \in M$, as $f \in O_n \leftrightarrow (\exists m \in \mathbb{N})\,\varphi(f(n), f(m))$. Then each O_n is open since $f \in O_n$ implies

$$(\exists k \in \mathbb{N})(\forall g \in M)\big(\overline{f}k =_{1^*} \overline{g}k \to g \in O_n\big) \tag{3.4}$$

by definition, implying $B\big(f, \frac{1}{2^{k+2}}\big) \subseteq O_n$ for $k \in \mathbb{N}$ as in (3.4).

In this light, the functional $\lambda f.\lambda n.(\mu m)\varphi(f(n),f(m))$ readily provides the function Y_n required for the R2-representation of O_n. That each O_n is dense in M follows by the assumption $\big(\forall x \in \mathbb{N}^\mathbb{N}\big)\big(\exists y \in \mathbb{N}^\mathbb{N}\big)\,\varphi(x,y)$. Hence, there is $h \in \bigcap_{n\in\mathbb{N}} O_n$ by (b), i.e., $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\,\varphi(h(n),h(m))$, by definition. Using $\mathsf{QF\text{-}AC}^{0,0}$, available in RCA_0, we obtain the required sequence and $\mathsf{QF\text{-}DC}^{1,1}$ follows. □

By Corollary 17, the Baire category theorem for R2-open sets does not imply $\mathsf{NIN}_{[0,1]}$; by contrast, the Baire category theorem for open sets (without R2-representation) readily implies Theorem 20 (c) and hence $\mathsf{NIN}_{[0,1]}$.

Finally, "a set has cardinality at most that of $2^\mathbb{N}$" can be expressed in various ways and we study the associated connections. We need the following definition.

Definition 22 (Sets of higher rank)**.** A set X of type σ objects is given by a characteristic function of type $\sigma \to \{0,1\}$. We assume such a set comes with an equivalence relation $=_X$ that satisfies $x =_\sigma y \to x =_X y$ for all $x, y \in X$.

Regarding the previous definition, we observe that real equality satisfies $x =_1 y \to x =_\mathbb{R} y$ for all $x, y \in \mathbb{R}$. As a further example, X could be $C\big(2^\mathbb{N}\big)$ and equality $F =_X G$ for $F^2, G^2 \in C\big(2^\mathbb{N}\big)$ is just $\big(\forall \sigma \in 2^{<\mathbb{N}}\big)\big(F(\sigma * 00\dots) =_0 G(\sigma * 00\dots)\big)\big)$, which can be defined in ACA_0^ω.

By the following theorem, it seems that $\mathsf{QF\text{-}AC}^{\sigma,1}$ is necessary to go from "surjection from $2^\mathbb{N}$ to X" to "injection from X to $2^\mathbb{N}$". Note that surjections and injections are defined relative to $=_X$.

[2] A set $O \subseteq M$ is *R2-open* if there is $Y\colon M \to \mathbb{R}$ with $x \in O \leftrightarrow [Y(x) >_\mathbb{R} 0 \wedge B(x, Y(x)) \subseteq O]$.

Theorem 23 ($\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{\sigma,1}$)**.** Let X be a set of type σ-objects and $F\colon 2^{\mathbb{N}} \to X$ be surjective, i.e., $(\forall x \in X)\big(\exists f \in 2^{\mathbb{N}}\big)(F(f) =_X x)$. Then there is $G^{\sigma\to 1}$ which is an injection from X to $2^{\mathbb{N}}$, i.e., $(\forall x, y \in X)(G(x) =_1 G(y) \to x =_X y)$.

Proof. Let $F\colon 2^{\mathbb{N}} \to X$ be surjective, i.e., $(\forall x^\sigma \in X)\big(\exists f \in 2^{\mathbb{N}}\big)(F(f) =_X x)$. Now apply $\mathsf{QF\text{-}AC}^{\sigma,1}$ to obtain $\Phi^{\sigma\to 1}$ with $\Phi(x) =_1 f$ in the previous. For $x, y \in X$, we have that $\Phi(x) =_1 \Phi(y)$ implies $F(\Phi(x)) =_\sigma F(\Phi(y))$ by extensionality, and hence $x =_X F(\Phi(x)) =_X F(\Phi(y)) =_X y$ by definition. □

We observe that Φ from the proof of the theorem need not satisfy extensionality relative to $=_X$, i.e., we may have that $x =_X y$ (but $x \neq_\sigma y$) and $\Phi(x) \neq_1 \Phi(y)$.

In conclusion, we have identified a couple of places in higher-order RM where $\mathsf{QF\text{-}AC}^{\sigma,1}$ is used in a non-trivial and perhaps essential way.

3.2 Streamlining known results

We discuss how quantifier-free choice as in $\mathsf{QF\text{-}AC}^{\sigma,1}$ can streamline known results.

First of all, the RM of basic topology proceeds smoothly in the presence of $\mathsf{QF\text{-}AC}^{1,1}$ (see [24]). In particular, various rather different versions of the Heine–Borel theorem and Lindelöf lemma are equivalent (see also [19, § 3.1.4]). In a nutshell, we have formulated *Cousin's lemma* from [6] as follows in, e.g., [16].

Principle 24 (HBU)**.** For $\Psi\colon [0,1] \to \mathbb{R}^+$, there are $x_0, \dots, x_k \in [0,1]$ such that $\bigcup_{i\leq k} B(x_i, \Psi(x_i))$ covers $[0,1]$.

The meaning of HBU is clear: the uncountable covering $\bigcup_{x\in[0,1]} B(x, \Psi(x))$ has a finite sub-covering. Unfortunately, the notion of covering in HBU is too restricted for the development of topology. In particular, the assumption that $x \in B(x, \Psi(x))$ is too strong. The following generalisation from [24] is equivalent assuming $\mathsf{QF\text{-}AC}^{1,1}$.

Principle 25 (HBT)**.** For $\psi\colon [0,1] \to \mathbb{R}^+ \cup \{0\}$ such that $(\forall x \in [0,1])(\exists y)(x \in B(y, \Psi(y)))$, there are $x_0, \dots, x_k \in [0,1]$ such that $\bigcup_{i\leq k} B(x_i, \psi(x_i))$ covers $[0,1]$.

Secondly, we have investigated the following explosive combinations.

(i) The combination of ACA_0^ω and the Lindelöf lemma for the Baire space proves $\Pi_1^1\text{-}\mathsf{CA}_0$ [18];
(ii) The combination of $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{2,1}$ and the Lindelöf lemma for the Baire space proves the Suslin functional as in (S^2) [18];
(iii) The combination of (S^2) with the Jordan decomposition theorem yields $\Pi_2^1\text{-}\mathsf{CA}_0$ [19].

Combing these items, $\Pi_2^1\text{-}\mathsf{CA}_0$ follows from of the combination of $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{2,1}$, the Lindelöf lemma for Baire space, and the Jordan decomposition theorem.

Thirdly, in the presence of $\mathsf{QF\text{-}AC}^{1,1}$, the functional S_k^2, which decides Σ_k^1-formulas, yields the Feferman–Sieg operators ν_k^2 from [4, p. 129], which returns a witness (if existent) to the Σ_k^1-formula at hand. Similarly, $\mathsf{QF\text{-}AC}^{2,1}$ suffices to prove the equivalence between Kleene's quantifier ($\exists^3$) and Hilbert-Bernays' operator ν from [10, p. 479].

Fourthly, filters and nets can characterise the same (topological) properties and one can switch between them (see [2]). A filter is, however, always of higher type than the associated net. To convert a (fourth-order) filter to a (third-order) net, as done in [2], $\mathsf{QF\text{-}AC}^{2,1}$ seems to suffice.

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