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Stationary list colorings

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Abstract. Komjáth [3] studied the list chromatic number of infinite graphs and introduced the notion of the restricted list chromatic number. For a graph $X = (V_X, E_X)$ and a cardinal κ, we say that X is restricted list colorable for κ ($\mathsf{RList}(X, \kappa)$) if, for any $L\colon V_X \to [\kappa]^\kappa$, there is some choice function c of L such that $c(v) \neq c(w)$ whenever $\{v, w\} \in E_X$. In this paper, we discuss its variation, the stationary list colorability for κ ($\mathsf{SList}(X, \kappa)$), which is obtained by replacing $[\kappa]^\kappa$ with the set of all stationary subsets of κ.

We compare the stationary list colorability with other coloring properties. Among other things, we prove that the stationary list colorability is essentially different from other coloring properties, including the restricted list colorability. We also prove that it is consistent that $\mathsf{RList}(X, \kappa)$ and $\mathsf{SList}(X, \kappa)$ do not imply $\mathsf{RList}(X, \lambda)$ and $\mathsf{SList}(X, \lambda)$ for $\lambda > \kappa$.

1 Introduction

Graph coloring is one of the central topics in combinatorics. Finite graph coloring has been studied extensively, and several invariants have been investigated, including the chromatic number, the coloring number, and the list chromatic number.

On the other hand, many natural extensions have been considered for infinite graphs as well. In particular, Erdős & Hajnal [1] studied the chromatic and the coloring numbers of infinite graphs widely. Let us recall those definitions. Let $X = (V_X, E_X)$ be an undirected simple graph, i.e., V_X is a set and $E_X \subseteq [V_X]^2$. A function c whose domain is V_X is *good* if, for all $v, w \in V_X$, $c(v) \neq c(w)$ holds whenever $\{v, w\} \in E_X$.

(1) The *coloring number* $\mathsf{Col}(X)$ is the least cardinal κ such that there is a well-order $\prec$ of V_X such that, for all $v \in V_X$, $\{w \in V_X \mid w \prec v \wedge \{v, w\} \in E_X\}$ has size $<\kappa$.
(2) The *chromatic number* $\mathsf{Chr}(X)$ is the least cardinal κ such that there is a good coloring $c\colon V_X \to \kappa$.

Kojman [2] studied the list chromatic number of infinite graphs.

(3) *The list chromatic number* $\mathsf{List}(X)$ is the least cardinal κ such that, for every function $L\colon V_X \to [\mathrm{Ord}]^\kappa$, there is some choice function c of L such that c is good.

In [2], the following was proved by using Shelah's revised GCH in ZFC theorem [5].

Fact 1 (Kojman, [2])**.** If $\mathsf{List}(X) \leq \kappa$, then $\mathsf{Col}(X) \leq \beth_\omega(\kappa)$.

Also, Kojman showed that, if GCH holds, then the fact above can be improved as follows.

Fact 2 (Kojman, [2])**.** Assume GCH. Suppose that $\mathsf{List}(X)$ is infinite. Then $\mathsf{Col}(X) \leq \mathsf{List}(X)^{++}$.

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Later, Komjáth [3] introduced the notion of the restricted list chromatic number:

(4) *The restricted list chromatic number* $\mathsf{List}^*(X)$ is the least cardinal κ such that, for every function $L\colon V_X \to [\kappa]^\kappa$, there is some choice function c of L such that c is good.

Komjáth [3] showed the consistency of $\mathsf{List}(X) = \mathsf{Col}(X)$ for every infinite graph X with $\mathsf{Col}(X) \geq \aleph_0$ and that of $\mathsf{List}(X) = \aleph_0$ if and only if $\mathsf{Chr}(X) = \aleph_0$ for every infinite graph X with $|V_X| = \aleph_1$. Also, Komjáth [3] proved the following, which contains an improvement of Fact 2:

Fact 3 (Komjáth, [3])**.** Assume GCH. Suppose that $\mathsf{List}(X)$ is infinite. Then $\mathsf{Col}(X) \leq \mathsf{List}(X)^+$. Also, if $\mathsf{List}^*(X)$ is regular, then $\mathsf{Col}(X) \leq \mathsf{List}^*(X)^+$.

Motivated by these works, we revisit and extend their approaches by introducing a new variant, called the stationary list coloring, where each vertex is assigned a stationary subset of a regular cardinal κ as its list of colors. In this paper, we study five coloring notions:

Definition 4. Let $X = (V_X, E_X)$ be a graph and κ be a cardinal.

(1) The property $\mathsf{Col}(X, \kappa)$ holds if there is a well-order $\prec$ of V_X such that, for all $v \in V_X$, the set $\{w \in V_X \mid w \prec v \wedge \{v, w\} \in E_X\}$ has size $< \kappa$.
(2) The property $\mathsf{List}(X, \kappa)$ holds if, for each $L\colon V_X \to [\mathrm{Ord}]^\kappa$, there is a choice function c of L such that c is a good coloring.
(3) The property $\mathsf{RList}(X, \kappa)$ holds if, for each $L\colon V_X \to [\kappa]^\kappa$, there is a choice function c of L such that c is a good coloring.
(4) Suppose κ is a regular cardinal. Let stat_κ be the set of all stationary subsets of κ. The property $\mathsf{SList}(X, \kappa)$ holds if, for each $L\colon V_X \to \mathsf{stat}_\kappa$, there is a choice function c of L such that c is a good coloring.
(5) The property $\mathsf{Chr}(X, \kappa)$ holds if there is a good coloring $c\colon V_X \to \kappa$.

For regular cardinals $\kappa < \lambda$, we can summarize the relationships among these properties as shown in Figure 1. See [3] for $\mathsf{Col}(X, \kappa) \to \mathsf{List}(X, \kappa)$. The implications $\mathsf{List}(X, \kappa) \to \mathsf{RList}(X, \kappa) \to \mathsf{SList}(X, \kappa)$ hold, since $\mathsf{stat}_\kappa \subseteq [\kappa]^\kappa \subseteq [\mathrm{Ord}]^\kappa$. The other implications are immediate from the definitions.

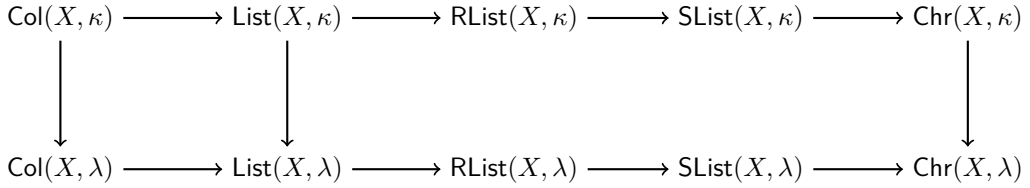


Figure 1: Diagram of variants of coloring properties.

Then the following questions arise naturally:

(1) Is $\mathsf{SList}(X, \kappa)$ essentially different from $\mathsf{RList}(X, \kappa)$ and $\mathsf{Chr}(X, \kappa)$?
(2) Can we prove analogues of Facts 1 and 3 for $\mathsf{SList}(X, \kappa)$?
(3) Are $\mathsf{RList}(X, -)$ and $\mathsf{SList}(X, -)$ monotonic? That is, if $\kappa < \lambda$, then does $\mathsf{RList}(X, \kappa)$ imply $\mathsf{RList}(X, \lambda)$, and does $\mathsf{SList}(X, \kappa)$ imply $\mathsf{SList}(X, \lambda)$?

In this paper, we answer these questions.

Definition 5. Let κ and λ be cardinals. The complete bipartite graph $K_{\kappa,\lambda}$ is defined as follows:

(1) $V_{K_{\kappa,\lambda}} = A \sqcup B$, where A and B are sets with $|A| = \kappa$ and $|B| = \lambda$.
(2) $E_{K_{\kappa,\lambda}} = \{\{a, b\} \mid a \in A, b \in B\}$.

Here, we can answer the questions above as follows:

Theorem 6 (see Corollary 9 and Theorems 10, 13, 14 and 25)**.** Let κ and λ be regular cardinals.

(1) (a) $\mathsf{Chr}(K_{2^\kappa,2^\kappa},\kappa)$ holds, but $\mathsf{SList}(K_{2^\kappa,2^\kappa},\kappa)$ fails.
(b) $\mathsf{SList}(K_{2^\kappa,\kappa},\kappa)$ holds, but $\mathsf{RList}(K_{2^\kappa,\kappa},\kappa)$ fails.
(2) If $\mathsf{SList}(X,\kappa)$ holds, then $\mathsf{Col}(X,\beth_\omega(\kappa))$ holds.
Furthermore, under GCH, the following stronger conclusions hold:
(a) $\mathsf{Col}(X,\kappa^{++})$ holds;
(b) moreover, if $\lambda \in I[\lambda]$ for all regular cardinals λ, then $\mathsf{Col}(X,\kappa^{+})$ holds.
(3) Assume GCH. Let $\kappa<\lambda$ be regular cardinals and $\mu>\lambda$ be a singular cardinal with cofinality κ. Define $\mathbb{P}=\mathrm{Fn}_{<\kappa}(\mu,2)$. Let G be a $\mathbb{P}$-generic filter over V. Then in $V[G]$,
(a) $\mathsf{RList}(K_{\mu,\mu},\kappa)$ holds, but $\mathsf{RList}(K_{\mu,\mu},\lambda)$ fails, and
(b) $\mathsf{SList}(K_{\mu,\mu},\kappa)$ holds, but $\mathsf{SList}(K_{\mu,\mu},\lambda)$ fails.
In other words, in $V[G]$, the monotonicities of RList and SList fail simultaneously.

The paper is organized as follows. In Section 2, we analyze the properties of stationary list colorings and prove that they are distinct from both the restricted and the ordinary list colorings. We also discuss an analogue of Komjáth's results. In Section 3, we present consistency results showing the failure of monotonicity for the stationary/restricted list-coloring properties. In Section 4, we conclude with several open questions and possible extensions of our results.

2 Analysis of stationary list coloring

First, we show that $\mathsf{SList}(X,\kappa)$ is different from $\mathsf{Chr}(X,\kappa)$. For later use, we state the claim in a more general form.

Definition 7. Let X be a set. The density of X ($\mathrm{den}(X,\subseteq)$) is the least cardinality of a dense subset of X. That is, $\mathrm{den}(X,\subseteq)\le\kappa$ if there is some $Y\subseteq X$ such that

(1) $|Y|\le\kappa$, and
(2) for every $x\in X$, there is some $y\in Y$ such that $y\subseteq x$.

Proposition 8. Let λ be a cardinal, and $X=K_{\lambda,\lambda}$. For a regular cardinal $\kappa\le\lambda$, if $\lambda\ge\mathrm{den}(\mathsf{stat}_\kappa,\subseteq)$ holds, then $\mathsf{SList}(X,\kappa)$ fails.

Proof. Let A,B be the two parts of V_X with $|A|=|B|=\lambda$. Let $\mathcal{S}\subseteq\mathsf{stat}_\kappa$ be a dense set of stationary sets in κ with $|\mathcal{S}|\le\lambda$. Take a list function L with $L``A=L``B=\mathcal{S}$. Let c be a choice function of L. It suffices to show that c is not good. Since $\mathcal{S}$ is dense in $(\mathsf{stat}_\kappa,\subseteq)$, $c``A$ intersects all stationary sets in $\kappa$, that is, $c``A$ contains a club subset in κ. Then there is some $S\in\mathcal{S}$ such that $S\subseteq c``A$ since $\mathcal{S}$ is dense in $(\mathsf{stat}_\kappa,\subseteq)$ and $\mathsf{club}_\kappa\subseteq\mathsf{stat}_\kappa$. Let $w\in B$ with $L(w)=S$. Then $c(w)\in L(w)=S\subseteq c``A$. Therefore, we can find a vertex $v\in A$ such that $c(v)=c(w)$. By the completeness of X, $\{v,w\}\in E_X$, and this implies that c is not a good coloring. □

Corollary 9. The property $\mathsf{Chr}(K_{2^\kappa,2^\kappa},\kappa)$ holds, but $\mathsf{SList}(K_{2^\kappa,2^\kappa},\kappa)$ fails.

Proof. Let $X=K_{2^\kappa,2^\kappa}$. Since X is a bipartite graph, even $\mathsf{Chr}(X,2)$ holds. Here, $2^\kappa\ge|\mathsf{stat}_\kappa|\ge\mathrm{den}(\mathsf{stat}_\kappa,\subseteq)$. Then, by Proposition 8, $\mathsf{SList}(X,\kappa)$ fails. □

The following theorem shows the difference between $\mathsf{SList}(X,\kappa)$ and $\mathsf{RList}(X,\kappa)$.

Theorem 10. The property $\mathsf{SList}(K_{2^\kappa,\kappa},\kappa)$ holds, but $\mathsf{RList}(K_{2^\kappa,\kappa},\kappa)$ fails.

Proof. Let $X=K_{2^\kappa,\kappa}$, and $V_X=A\sqcup B$ be the bipartite graph with $|A|=\kappa$ and $|B|=2^\kappa$. First, we show that $\mathsf{SList}(X,\kappa)$ holds. To prove this, take an arbitrary list function $L\colon V_X\to\mathsf{stat}_\kappa$ and construct a good

coloring c choosing from L. We identify A with κ. Since κ is regular, we can choose $\langle c(i) \in L(i) \mid i < \kappa\rangle$ and $\langle \gamma_i < \kappa \mid i < \kappa\rangle$ as follows:

(1) For any $i < \kappa$, $\gamma_i < c(i) < \gamma_{i+1}$.
(2) If $i < \kappa$ is a limit ordinal, then $\sup_{j<i} \gamma_j = \gamma_i$.

Then $\{\gamma_i \mid i < \kappa\}$ is a club in κ and $\{c(i) \mid i < \kappa\} \cap \{\gamma_i \mid i < \kappa\} = \varnothing$. Since $L(v)$ is stationary in κ for every $v \in V_X$, $L(v) \cap \{\gamma_i \mid i < \kappa\} \neq \varnothing$, and we can take $c(v) \in L(v) \cap \{\gamma_i \mid i < \kappa\}$ for all $v \in B$. Thus c is a good coloring.

Next, we prove that $\mathsf{RList}(X, \kappa)$ fails. To prove this, we construct a list function $L\colon V_X \to [\kappa]^\kappa$ such that, for any choice function c of L, c is not good. We identify A and B with κ and $[\kappa]^\kappa$, respectively. Define the list function L by

$$L(v) = \begin{cases} \kappa \backslash v & \text{if } v \in A = \kappa, \\ v & \text{if } v \in B = [\kappa]^\kappa. \end{cases}$$

Let c be a choice function of L, and we show that c is not good. Let $w = c \text{“ } A$. By the definition of c and the regularity of κ, $w \in [\kappa]^\kappa = B$. Since $c(w) \in L(w) = w = c \text{“ } A$, there is some $v \in A$ such that $c(w) = c(v)$. By the completeness of X, $\{v, w\} \in E_X$. Therefore, c is not a good coloring. □

We can apply Corollary 9 to obtain an analogue of Fact 1. We first prepare some notions and facts.

Definition 11. Let X and Y be graphs, then X is *Y-free* if X does not contain any subgraph isomorphic to Y.

In order to prove Fact 1, Kojman showed the following:

Fact 12. Let κ be an infinite cardinal, and $\lambda \geq \beth_\omega(\kappa)$. Then $\mathsf{Col}(X, \lambda)$ holds for every K_{κ,λ^+}-free graph X.

We have already shown that $\mathsf{SList}(K_{2^\kappa,2^\kappa}, \kappa)$ fails for every regular cardinal κ. Now we obtain the following.

Theorem 13. Let κ be a regular cardinal. If $\mathsf{SList}(X, \kappa)$ holds, then $\mathsf{Col}(X, \beth_\omega(\kappa))$ holds.

Proof. We suppose $\mathsf{SList}(X, \kappa)$ and prove $\mathsf{Col}(X, \beth_\omega(\kappa))$. The property $\mathsf{SList}(K_{2^\kappa,2^\kappa}, \kappa)$ fails by Corollary 9. Thus, since $\mathsf{SList}(X, \kappa)$ holds, X must be $K_{2^\kappa,2^\kappa}$-free. Therefore, X is also $K_{2^\kappa,(\beth_\omega(2^\kappa))^+}$-free since $2^\kappa \leq \beth_\omega(2^\kappa)$. By Fact 12, $\mathsf{Col}(X, \beth_\omega(2^\kappa))$ holds. Since $\beth_\omega(\kappa) = \beth_\omega(2^\kappa)$, $\mathsf{Col}(X, \beth_\omega(2^\kappa))$ is equivalent to $\mathsf{Col}(X, \beth_\omega(\kappa))$. □

We note that Theorem 13 is an improvement of Fact 1 for regular κ since $\mathsf{List}(X, \kappa)$ implies $\mathsf{SList}(X, \kappa)$. Next, we prove an analogue of Fact 3. More concretely, we prove the following.

Theorem 14. Assume GCH. Let κ be a regular cardinal. If $\mathsf{SList}(X, \kappa)$ holds, then

(1) $\mathsf{Col}(X, \kappa^{++})$ holds;
(2) moreover, if $\lambda \in I[\lambda]$ for all regular cardinals λ, then $\mathsf{Col}(X, \kappa^+)$ holds.

By Theorem 10, $\mathsf{SList}(X, \kappa)$ does not imply $\mathsf{Col}(X, \kappa)$, or even $\mathsf{RList}(X, \kappa)$, in general. Therefore, at least Theorem 14 (2) is optimal. In order to prove Theorem 14, we need to prepare several properties and facts. First, we recall the ideal $I[\lambda]$ for a regular cardinal λ.

Definition 15. Define $I[\lambda]$ as the set of all $S \in \mathcal{P}(\lambda)$ such that there exist a sequence $\langle b_\xi \mid \xi < \lambda\rangle$ and a club $C \subseteq \lambda$ such that

(1) each b_ξ is a bounded subset of λ and
(2) for all $\gamma \in C \cap S$, there is some $c_\gamma \subseteq \gamma$ such that
  (a) the set c_γ is cofinal in γ,
  (b) $\mathrm{otp}(c_\gamma) = \mathrm{cf}(\gamma)$, and
  (c) for all $\delta < \gamma$, there is some $\eta < \gamma$ such that $c_\gamma \cap \delta = b_\eta$.

In order to prove Fact 3, Komjáth [3] introduced some specific graphs as follows. Here, let $N(v)$ denote the set of all neighbors of v for $v \in V_X$. That is, $N(v) = \{w \in V_X \mid \{v, w\} \in E_X\}$.

Definition 16. Let κ be a cardinal.

(1) We say that a bipartite graph X is *κ-type 1* if, for some cardinal $\lambda \geq \kappa$, we have
   (a) the bipartition (A, B) satisfies $|A| = \lambda$ and $|B| = \lambda^+$, and
   (b) $|N(b)| \geq \kappa$ holds for all $b \in B$.

(2) We say that a graph X is *κ-type 2* if there is some regular cardinal $\lambda > \kappa$ such that X is isomorphic to the following graph Y:
   (a) $V_Y = \lambda$, and
   (b) $T_Y = \left\{\xi \in E^{\lambda}_{\mathrm{cf}(\kappa)} \;\middle|\; \mathrm{otp}(N(\xi) \cap \xi) = \kappa \wedge N(\xi) \cap \xi \text{ is cofinal in } \xi\right\}$ is stationary in λ.

Komjáth [3] showed a necessary and sufficient condition for $\mathsf{Col}(X, \kappa)$.

Fact 17. Let X be an infinite graph and κ be an infinite cardinal.

(1) If $\mathsf{Col}(X, \kappa)$ fails, then X has a subgraph Y such that Y is κ-type 1 or 2.
(2) If $\mathsf{Col}(X, \kappa^+)$ fails, then X has a subgraph Y such that Y is κ-type 1.

In the proof of Fact 3, Komjáth actually proved the contrapositive. More concretely, in [3], the following is proved.

Fact 18. Assume GCH. Let κ be a cardinal and X be a κ-type 1 graph. Then

(1) $\mathsf{List}(X, \kappa)$ fails, and
(2) if κ is regular, then $\mathsf{RList}(X, \kappa)$ fails.

Combining Facts 17 and 18, we obtain that, for a cardinal κ, the failure of $\mathsf{Col}(X, \kappa^+)$ implies the failure of $\mathsf{List}(X, \kappa)$, and also that of $\mathsf{RList}(X, \kappa)$ if κ is regular. This is the outline of Komjáth's proof.

Our strategy for the proof of Theorem 14 is based on this argument. That is, we show that $\mathsf{SList}(Y, \kappa)$ fails for each κ^+-type 1 graph Y under GCH, and for each κ^+-type 2 graph Y under GCH and the approachability assumption.

Lemma 19. Assume GCH. If X is κ^+-type 1, then $\mathsf{SList}(X, \kappa)$ fails.

Proof. Let $V_X = A \sqcup B$ be the bipartition of V_X with $|A| = \lambda$ and $|B| = \lambda^+$ for some cardinal λ.

We construct a list function $L\colon A \sqcup B \to \mathsf{stat}_\kappa$ that has no good coloring function. We may assume that $A = \lambda$. By taking a subgraph of X, if necessary, we can ensure that this λ is minimal with respect to the conditions in Definition 16. That is, if $A' \subseteq A$, $B' \subseteq B$, $|A'|^+ = |B'|$, and $|N(b) \cap A'| \geq \kappa^+$ for all $b \in B$, then $|A'| = \lambda$. Also, by shrinking the edges, we may assume that $|N(b)| = \kappa^+$ for all $b \in B$.

Claim 20. The equality $\mathrm{cf}(\lambda) = \kappa^+$ holds.

Proof of Claim 20. For a contradiction, suppose that $\mathrm{cf}(\lambda) \neq \kappa^+$. Note that $\kappa^+ < \lambda$. For each $b \in B$, there is some $\xi_b < \lambda$ such that $|N(b) \cap \xi_b| = \kappa^+$. By the pigeonhole principle, there is some $\xi < \lambda$ such that $B_0 = \{b \in B \mid \xi_b = \xi\}$ has size λ^+. Since GCH holds, $\left|[\xi]^{\kappa^+}\right| \leq \lambda$. Therefore, we can take $B_1 \subseteq B_0$ with $|B_1| = \lambda^+$ such that $A' = N(b) \cap \xi$ is constant on $b \in B_1$. Let $B' \subseteq B_1$ with $|B'| = \kappa^{++}$. Then, $X \restriction (A' \sqcup B')$ is a complete bipartite subgraph, and $\kappa^{++} = |A'|^+ = |B'|$. However, this contradicts the minimality of λ since $\kappa^+ < \lambda$. ∎

Take a cofinal sequence $\langle \lambda_\sigma \mid \sigma < \kappa^+ \rangle$ with $\lambda_0 = 0$. Since GCH holds, there is a sequence $\langle C_\sigma \mid \sigma < \kappa^+ \rangle$ of clubs in κ such that

(1) if $\sigma \leq \tau$, then $C_\tau \subseteq^* C_\sigma$, and
(2) for each club C, there is some $\sigma < \kappa^+$ such that $C_\tau \subseteq^* C$ for all $\tau \geq \sigma$,

where $S \subseteq^* S'$ means $S \backslash S'$ is bounded in κ for $S, S' \in \mathcal{P}(\kappa)$.

For $\xi \in [\lambda_\sigma, \lambda_{\sigma+1}) \subseteq \lambda = A$, we define $F(\xi) = C_\sigma$.

Claim 21. For $b \in B$, there is an $i_b < \kappa$ such that, for every club C in κ, there exists some $\xi \in N(b)$ such that $F(\xi)\backslash(i_b + 1) \subseteq C$ holds.

Proof of Claim 21. Take an arbitrary $b \in B$. Suppose, for contradiction, that there is some $C_i \subseteq \kappa$ for each $i < \kappa$ such that

(1) the set C_i is a club in κ, and
(2) for every $\xi \in N(b)$, we have $F(\xi)\backslash(i+1) \nsubseteq C_i$.

Let $D = \bigtriangleup_{i<\kappa} C_i$. Since D is a club in κ, there is some $\xi \in N(b)$ and $i < \kappa$ such that $F(\xi)\backslash(i+1) \subseteq D$. By the definition of D, $D\backslash(i+1) \subseteq C_i$. Therefore, $F(\xi)\backslash(i+1) \subseteq D\backslash(i+1) \subseteq C_i$, and this contradicts the choice of C_i. ■

By the pigeonhole principle, there exists an $i < \kappa$ such that $B' = \{b \in B \mid i_b = i\}$ has size λ^+. We define the list function L as follows. For $\xi \in A$, we define $L(\xi) = F(\xi)\backslash(i+1)$. Since GCH holds, we can identify B' with the set $\prod_{\xi\in A} L(\xi)$. Here, for $g \in B'$, define $L(g) = \{g(\xi) \mid \xi \in N(g)\}$, and for $b \in B\backslash B'$, define $L(b) = \kappa$. We show that L is as desired. To prove this, we have to check that L has no good coloring function and $L(g)$ is stationary in κ for all $g \in B'$.

First, we show that L does not have a good coloring. Let c be a choice function of L and $g = c \restriction A$. Then, $g \in \prod_{\xi\in A} L(\xi) = B'$. By the definition of L, $c(g) \in L(g) = \{g(\xi) \mid \xi \in N(g)\}$. Therefore, there is some $\xi \in N(g)$ such that $c(g) = g(\xi) = c(\xi)$. Since $\xi \in N(g)$, c is not good.

Next, we prove that $L(g)$ is stationary in κ for all $g \in B'$. Take an arbitrary $g \in B'$ and a club set C in κ. It suffices to show that $L(g) \cap C \neq \varnothing$. By the choice of i and B', there is a $\xi \in N(g)$ such that $L(\xi) = F(\xi)\backslash(i+1) \subseteq C$. Therefore, $g(\xi) \in L(g) \cap C \neq \varnothing$, so we are done. □

Now we can prove Theorem 14 (1).

Proof of Theorem 14 (1). Let X be an infinite graph. We show the contrapositive; that is, we assume the failure of $\mathsf{Col}(X, \kappa^{++})$ and show the negation of $\mathsf{SList}(X, \kappa)$. By Fact 17 (2), X has a subgraph Y such that Y is κ^+-type 1. Then, by Lemma 19, $\mathsf{SList}(Y, \kappa)$ fails. This completes the proof. □

Next, we prove Theorem 14 (2). In order to prove this, by Fact 17 (1) and Lemma 19, it suffices to show that, for every κ^+-type 2 graph X, $\mathsf{SList}(X, \kappa)$ fails assuming that GCH holds and $\lambda \in I[\lambda]$ for all λ. Take such a graph X, and let $V_X = \lambda$ and T_X be the set defined in Definition 16.

Lemma 22. Assume GCH. Suppose that, for all sufficiently large regular cardinals ϑ, there is an elementary substructure $M \prec \langle \mathcal{H}_\vartheta, \in, X, \kappa\rangle$ with $|M| < \lambda$ such that

(1) $M \cap \lambda \in T_X$, and
(2) there is some $x \subseteq N(M \cap \lambda) \cap (M \cap \lambda)$ with $|x| = \kappa^+$ such that $x \cap \gamma \in M$ for all $\gamma < M \cap \lambda$.

Then, $\mathsf{SList}(X, \kappa)$ fails.

Proof. Let M and x be witnesses and $\xi^* = M \cap \lambda$. For $\sigma < \kappa^+$, we define x_σ as the initial segment of x with $\mathrm{otp}(x_\sigma) = \sigma$, and S_σ as the set of all $\xi \in T_X\backslash(\sup(x_\sigma) + 1)$ such that $x_\sigma \subseteq N(\xi)$ holds. Since $X, x_\sigma \in M$, $S_\sigma \in M$. Then for each $\sigma < \kappa^+$, S_σ is a stationary subset of λ since $\xi^* \in S_\sigma$. Since $\langle S_\sigma \mid \sigma < \kappa^+\rangle$ is a $\subseteq$-decreasing continuous sequence of stationary sets with $\bigcap_{\sigma<\kappa^+} S_\sigma = \{\xi^*\}$, we can take $A \in [\kappa^+]^{\kappa^+}$ such that $T_\sigma = (S_\sigma\backslash S_{\sigma+1})\backslash\xi^*$ is unbounded in λ for all $\sigma \in A$.

Let club_κ be the set of all club subsets in κ. Take a list function $L\colon V_X \to \mathsf{stat}_\kappa$ satisfying $L``x = \mathsf{club}_\kappa$, $L``T_\sigma = \mathsf{stat}_\kappa$ for each $\sigma \in A$, and $L(v) = \kappa$ for all other vertices v. We show that L is as desired. In order to prove this, take an arbitrary choice function c of L and show that c is not good.

Since $L``x = \mathsf{club}_\kappa$, $c``x$ intersects every club set of κ, that is, $c``x$ is stationary. Then, since $\mathrm{otp}(x) = \kappa^+$, there is some $\sigma \in A$ such that $c``x_\sigma = c``x$. By the definition of $L$, there is some $\xi \in T_\sigma$ such that $L(\xi) = c``x_\sigma$. Thus there is an $\eta \in x_\sigma$ such that $c(\eta) = c(\xi)$. Since $\xi \in T_\sigma \subseteq S_\sigma$, $\{\eta, \xi\} \in E_X$. Therefore, c is not a good function. □

Lemma 23. Assume GCH. If $\lambda \in I[\lambda]$ for every cardinal λ and X is κ^+-type 2, then $\mathsf{SList}(X, \kappa)$ fails.

Proof. Let $V_X = \lambda$ and $T_X \subseteq E^\lambda_{\kappa^+}$ be the set defined in Definition 16.

Case 1: λ is a limit cardinal, or $\lambda = \zeta^+$ for some cardinal ζ with $\mathrm{cf}(\zeta) \neq \kappa$. It suffices to show that the assumptions in Lemma 22 hold. Take a sufficiently large regular cardinal ϑ. By the cardinal arithmetic assumptions, we can find $M \prec \langle \mathcal{H}_\vartheta, \in, X, \kappa\rangle$ with $|M| < \lambda$, $M^\kappa \subseteq M$, and $M \cap \lambda \in T_X$. Let $\xi^* = M \cap \lambda$. Note that $\mathrm{cf}(\xi^*) = \kappa^+$ and $N(\xi^*)$ is cofinal in ξ^* with $\mathrm{otp}(N(\xi^*)) = \kappa^+$. We show that $x = N(\xi^*)$ is a witness. For $\gamma < \xi^*$, $x \cap \gamma \in M$ holds since $M^\kappa \subseteq M$ and $|x \cap \gamma| \leq \kappa$. Thus, we are done.

Case 2: $\lambda = \zeta^+$ for some ζ with $\mathrm{cf}(\zeta) = \kappa$. It suffices to show that the assumptions in Lemma 22 hold. In order to prove this, take a sufficiently large regular cardinal ϑ and find witnesses $M \prec \langle \mathcal{H}_\vartheta, \in, X, \kappa\rangle$ and $x \subseteq N(M \cap \lambda)$. Since $\lambda \in I[\lambda]$, we can take $\langle b_\xi \mid \xi < \lambda\rangle$ and $C \subseteq \lambda$ such that

(1) each b_ξ is a bounded subset of λ, and
(2) for all $\gamma \in C$, there is some $c_\gamma \subseteq \gamma$ such that
  (a) the set c_γ is cofinal in γ,
  (b) $\mathrm{otp}(c_\gamma) = \mathrm{cf}(\gamma)$, and
  (c) for all $\delta < \gamma$, there is some $\eta < \gamma$ such that $c_\gamma \cap \delta = b_\eta$.

Take an elementary substructure M of $\langle \mathcal{H}_\vartheta, \in, X, \kappa, \zeta\rangle$ such that $|M| < \lambda$, $\langle b_\xi \mid \xi < \lambda\rangle$, $C \in M$, $\zeta \subseteq M$, and $M \cap \lambda \in T_X \subseteq E^\lambda_{\kappa^+}$. Let $\xi^* = M \cap \lambda$. Note that $\mathrm{cf}(\xi^*) = \kappa^+$.

Since $C \in M$ and C is a club, $C \cap \xi^*$ is cofinal in ξ^*, and hence $\xi^* \in C$. Since $\lambda \in I[\lambda]$, there is some cofinal subset $c \subseteq \xi^*$ such that $\mathrm{otp}(c) = \mathrm{cf}(\xi^*) = \kappa^+$ and for all $\delta < \xi^*$, there is some $\eta < \xi^*$ such that $c \cap \delta = b_\eta$. Since $\langle b_\xi \mid \xi < \lambda\rangle \in M$ and $\eta < \xi^* = M \cap \lambda$, $c \cap \delta = b_\eta \in M$ for all $\delta < \xi^*$. Take such a cofinal subset c.

Let $\langle \pi_\xi \mid \xi < \lambda\rangle$ be a sequence in M such that π_ξ is a surjection from ζ to ξ for all $\xi < \lambda$. Take a sequence $\langle \zeta_i \mid i < \kappa\rangle \in M$ cofinal in ζ with $\zeta_0 > \kappa$. For every $i < \kappa$, define X_i as follows:

$$X_i = \left\{\eta < \xi^* \;\middle|\; \pi_{\min(c\setminus(\eta+1))}``\zeta_i \ni \eta\right\}.$$

Since $\mathrm{cf}(\xi^*) = \kappa^+$ and $\langle X_i \mid i < \kappa\rangle$ is a $\subseteq$-increasing sequence such that $\bigcup_{i<\kappa} X_i = \xi^*$, we can find an $i < \kappa$ such that $X_i \cap N(\xi^*)$ is unbounded in ξ^*. Take such an $i < \kappa$ and show that $x = X_i \cap N(\xi^*)$ is as desired. That is, we show that $x \cap \gamma \in M$ for all $\gamma < \xi^*$.

For every $\gamma < \xi^*$, $X_i \cap \gamma = \left\{\eta < \gamma \mid \pi_{\min(c\setminus(\eta+1))}``\zeta_i \ni \eta\right\}$ can be defined by using parameters in M since $\min(c\setminus(\eta+1)) \in M$, so $X_i \cap \gamma \in M$. Here, $|X_i \cap \gamma| \leq |X_i| \leq \zeta_i$. Therefore, by using GCH and $\zeta \subseteq M$, $X_i \cap \gamma \cap N(\xi^*) \in M$ for all $\gamma < \xi^*$.

In any case, we proved the negation of $\mathsf{SList}(X, \kappa)$. This completes the proof. □

Combining Lemmata 19 and 23, we obtain Theorem 14 (2).

Proof of Theorem 14 (2). Let X be an infinite graph. Suppose that $\mathsf{Col}(X, \kappa^+)$ fails, then we show the negation of $\mathsf{SList}(X, \kappa)$. By Fact 17 (1), X has a subgraph Y such that Y is κ^+-type 1 or 2. If Y is type 1, Lemma 19 implies the failure of $\mathsf{SList}(X, \kappa)$. On the other hand, if Y is type 2, Lemma 23 shows the negation of $\mathsf{SList}(X, \kappa)$. This completes the proof. □

Theorem 14 implies the following immediately.

Corollary 24. Assume GCH. Let κ be a regular cardinal. Then

(1) $\mathsf{SList}(X, \kappa)$ implies $\mathsf{SList}(X, \kappa^{++})$, and
(2) if $\lambda \in I[\lambda]$ for all regular cardinals λ, then $\mathsf{SList}(X, \kappa)$ implies $\mathsf{SList}(X, \kappa^+)$.

3 Anti-monotonicity of coloring properties

In this section, we show consistency results about anti-monotonicity of coloring properties. Our proof is based on the singular-cardinal approach of Raghavan & Shelah [4] and Usuba [6]. The aim of this section is to prove the following result.

Theorem 25. Assume GCH. Let $\kappa < \lambda$ be regular cardinals. Let $\mu > \lambda$ be a singular cardinal with cofinality κ. Define $\mathbb{P} = \mathrm{Fn}_{<\kappa}(\mu, 2)$. Let G be a $\mathbb{P}$-generic filter over V. Then, in $V[G]$, $\mathsf{RList}(K_{\mu,\mu}, \kappa)$ holds, but $\mathsf{SList}(K_{\mu,\mu}, \lambda)$ fails.

We first introduce a generalization of the reaping number $\mathfrak{r}$.

Definition 26. Let κ be a regular cardinal.

(1) For $A, B \in [\kappa]^\kappa$, we say that A κ-splits B if $|B \cap A| = |B \backslash A| = \kappa$.
(2) For $\mathcal{R} \subseteq [\kappa]^\kappa$, we say that $\mathcal{R}$ is a κ-reaping family if, for any $A \in [\kappa]^\kappa$, there is some $B \in \mathcal{R}$ such that A does not κ-split B.
(3) Define $\mathfrak{r}_\kappa = \min\{|\mathcal{R}| \mid \mathcal{R} \text{ is a } \kappa\text{-reaping family}\}$.

For example, $\mathfrak{r}$ coincides with $\mathfrak{r}_\omega$. The next proposition states that $\lambda < \mathfrak{r}_\kappa$ is characterized precisely by $\mathsf{RList}(K_{\lambda,\lambda}, \kappa)$.

Proposition 27. Let λ be a cardinal, and $X = K_{\lambda,\lambda}$. For a regular cardinal $\kappa \leq \lambda$, then the following are equivalent.

(1) $\lambda < \mathfrak{r}_\kappa$.
(2) $\mathsf{RList}(X, \kappa)$.

Proof. Let A, B be the two parts of V_X with $|A| = |B| = \lambda$.

(1) $\Rightarrow$ (2). Let $L\colon A \sqcup B \to [\kappa]^\kappa$ be a list function. It suffices to construct a choice function c of L such that c is good. Let $\mathcal{F} = \mathrm{ran}(L) \subseteq [\kappa]^\kappa$. Then $|\mathcal{F}| \leq \lambda < \mathfrak{r}_\kappa$. By the definition of $\mathfrak{r}_\kappa$, there is some $S \in [\kappa]^\kappa$ such that S κ-splits all elements of $\mathcal{F}$. In particular, for all $v \in V_X$, $L(v) \cap S$ and $L(v) \backslash S$ are not empty. Define c as follows:

$$c(v) \in \begin{cases} L(v) \cap S & \text{if } v \in A, \text{ and} \\ L(v) \backslash S & \text{if } v \in B. \end{cases}$$

Thus c is a good coloring.

(2) $\Rightarrow$ (1). We show the contrapositive, that is, we suppose $\lambda \geq \mathfrak{r}_\kappa$ and show the negation of $\mathsf{RList}(X, \kappa)$. Let $\mathcal{R}$ be a reaping family with $|\mathcal{R}| = \mathfrak{r}_\kappa$. We show that there is a list function $L\colon A \sqcup B \to [\kappa]^\kappa$ such that no choice function c of L is good. Take a list function L satisfying $L \text{“ } A = L \text{“ } B = \{X \backslash i \mid X \in \mathcal{R}, i < \kappa\}$. Let c be an arbitrary choice function of L. Let $S_0 = c \text{“ } A$ and $S_1 = c \text{“ } B$. By the definition of L, $|S_n \cap S| = \kappa$ for every $S \in \mathcal{R}$ and $n < 2$. Since $\mathcal{R}$ is a reaping family, we can take $S \in \mathcal{R}$ such that $|S \backslash S_0| < \kappa$. Since $|S \cap S_1| = \kappa$ and

$$S \cap S_1 \subseteq (S_0 \cap S_1) \cup (S \backslash S_0) \cap S_1,$$

we have $S_0 \cap S_1 \neq \varnothing$. Let $\gamma \in S_0 \cap S_1$, $v \in A$, and $w \in B$ such that $c(v) = c(w) = \gamma$. Since X is the complete bipartite graph, $\{v, w\} \in E_X$. Therefore, c is not good. □

Proof of Theorem 25. Take a continuous increasing sequence $\langle \mu_i \mid i < \kappa \rangle$ cofinal in μ. By Propositions 8 and 27, it suffices to show that $V[G] \vDash$ “$\mathrm{den}(\mathsf{stat}_\lambda, \subseteq) \leq \mu < \mathfrak{r}_\kappa$”. Note that $\mathbb{P}$ is $<\kappa^+$-cc.

Claim 28. In $V[G]$, $\mathrm{den}(\mathsf{stat}_\lambda, \subseteq) \leq \mu$ holds.

Proof of Claim 28. By counting nice names, we can show that $\left|(\mathcal{P}(\lambda))^{V[G \restriction \mu_i]}\right| \leq \mu_i$ holds. Thus it suffices to show that $\bigcup_{i<\kappa} (\mathcal{P}(\lambda))^{V[G \restriction \mu_i]} \cap \mathsf{stat}_\lambda^{V[G]}$ is dense in $(\mathsf{stat}_\lambda, \subseteq)$ in $V[G]$.

Let $\dot{S} \in V^{\mathbb{P}}$ be a nice name for a subset of λ and $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{S}$ is stationary in λ". We need to find a condition $q \leq p$, $i < \kappa$, and a name $\dot{T} \in V^{\mathrm{Fn}_{<\kappa}(\mu_i,2)}$ such that $q \Vdash$ "$\dot{T}$ is stationary and $\dot{T} \subseteq \dot{S}$". Let $\dot{S} = \bigcup_{\xi<\lambda} \{\check{\xi}\} \times A_\xi$ where A_ξ is an antichain in $\mathbb{P}$. For $i < \kappa$, define

$$\dot{S}_i = \bigcup_{\xi<\lambda} \{\check{\xi}\} \times (A_\xi \cap \mathrm{Fn}_{<\kappa}(\mu_i, 2)) \in V^{\mathrm{Fn}_{<\kappa}(\mu_i,2)}.$$

Since $p \Vdash$ "$\dot{S} = \bigcup_{i<\kappa} \dot{S}_i$" and $\kappa < \lambda$, there are some $q \leq p$ and $i < \kappa$ such that $q \Vdash$ "$\dot{S}_i$ is stationary in λ". The name $\dot{T} = \dot{S}_i$ is as desired. ■

Claim 29. In $V[G]$, $\mu < \mathfrak{r}_\kappa$ holds.

Proof of Claim 29. Let $p \in \mathbb{P}$ and $\{\dot{X}_\alpha \mid \alpha < \mu\} \subseteq V^{\mathbb{P}}$ be a family of nice names for elements of $[\kappa]^\kappa$. We need to find a name $\dot{X}$ for an element of $[\kappa]^\kappa$ such that $p \Vdash$ "$\dot{X}$ κ-splits all elements of $\{\dot{X}_\alpha \mid \alpha < \mu\}$". Let $\dot{X}_\alpha = \bigcup_{i<\kappa} \{\check{i}\} \times A^\alpha_i$ where A^α_i is an antichain in $\mathbb{P}$. For $i < \kappa$, define $S_i = \bigcup \{\mathrm{supp}(q) \mid q \in A^\alpha_j,\ j < \kappa,\ \alpha < \mu_i\}$. Since $\mathbb{P}$ is $<\kappa^+$-cc, $|S_i| \leq \mu_i \cdot \kappa^+ < \mu$. Thus we can take $\gamma_i < \mu$ such that $\gamma_i \notin S_i$ for $i < \kappa$. Let

$$\dot{X} = \{(\check{i}, \{(\gamma_i, 1)\}) \mid i < \kappa\} \in V^{\mathbb{P}}.$$

We show that $\dot{X}$ is as desired.

Take $p' \leq p$, $\alpha < \mu$, and $i < \kappa$. It suffices to show that there are some $q_0, q_1 \leq p'$ and $j \geq i$ such that $q_0 \Vdash$ "$\check{j} \in \dot{X}_\alpha \backslash \dot{X}$" and $q_1 \Vdash$ "$\check{j} \in \dot{X}_\alpha \cap \dot{X}$". We can take $j \geq i$ such that $\mu_j > \alpha$, $p' \nVdash$ "$j \notin \dot{X}_\alpha$", and $\gamma_j \notin \mathrm{supp}(p')$ since $p \Vdash$ "$\dot{X}_\alpha \in [\kappa]^\kappa$" and $|\mathrm{supp}(p')| < \kappa$. Then there is some $a \in A^\alpha_j$ such that $a \parallel p'$. Since $\gamma_j \notin \mathrm{supp}(p') \cup S_j$, both $\{(\gamma_j, 0)\}$ and $\{(\gamma_j, 1)\}$ are compatible with a and p'. Here

$$q_0 = a \cup p' \cup \{(\gamma_j, 0)\} \Vdash \text{"}\check{j} \in \dot{X}_\alpha \backslash \dot{X}\text{"}, \text{ and}$$
$$q_1 = a \cup p' \cup \{(\gamma_j, 1)\} \Vdash \text{"}\check{j} \in \dot{X}_\alpha \cap \dot{X}\text{"}.$$

Therefore, $p \Vdash$ "$\dot{X}$ κ-splits $\dot{X}_\alpha$" for all $\alpha < \mu$. ■

This completes the proof. □

Since $\mathsf{RList}(X, \nu)$ implies $\mathsf{SList}(X, \nu)$ for any graph X and regular cardinal ν, Theorem 25 implies the following directly.

Corollary 30. Let $V[G]$ be the forcing extension from Theorem 25. Then in $V[G]$,

(1) $\mathsf{RList}(K_{\mu,\mu}, \kappa)$ holds, but $\mathsf{RList}(K_{\mu,\mu}, \lambda)$ fails, and
(2) $\mathsf{SList}(K_{\mu,\mu}, \kappa)$ holds, but $\mathsf{SList}(K_{\mu,\mu}, \lambda)$ fails.

In other words, in $V[G]$, the monotonicities of $\mathsf{RList}(K_{\mu,\mu}, -)$ and $\mathsf{SList}(K_{\mu,\mu}, -)$ fail simultaneously.

4 Questions

In Corollary 9 and Theorem 10, we constructed graphs witnessing the fact that $\mathsf{SList}(X, \kappa)$ is different from $\mathsf{RList}(X, \kappa)$ and $\mathsf{Chr}(X, \kappa)$. On the other hand, Komjáth [3] proved that $\mathsf{RList}(X, \kappa)$ consistently differs from $\mathsf{List}(X, \kappa)$. Then the following question arises:

Question 31. Can we prove that $\mathsf{RList}(X, \kappa)$ is different from $\mathsf{List}(X, \kappa)$ in ZFC?

Next, we do not know whether the assumption of Theorem 14 is optimal or not. That is:

Question 32. Under GCH, can we prove that $\mathsf{SList}(X, \kappa)$ implies $\mathsf{Col}(X, \kappa^+)$ without any assumption with respect to regular κ?

In Section 3, we constructed the model that satisfies $\mathsf{RList}(X, \kappa)$ and the negation of $\mathsf{SList}(X, \lambda)$ for some graph X and regular cardinals $\kappa < \lambda$. In this model, actually, we can prove that $\mathfrak{r}_\lambda$ is singular and $\mathfrak{r}_\lambda < \mathfrak{r}_\kappa$ by a

method similar to that of Theorem 25. Can we show this anti-monotonicity of reaping numbers by some small forcings? More concretely:

Question 33. Is it consistent that $\mathfrak{r} = \omega_3 > \omega_2 = \mathfrak{r}_{\omega_1}$?

In ZFC, $\mathfrak{r}_\mu \geq \mathfrak{r}_{\mathrm{cf}(\mu)}$ holds for singular μ. Can we separate these invariants?

Question 34. Is it consistent that $\mathfrak{r}_\mu > \mathfrak{r}_{\mathrm{cf}(\mu)}$ for singular μ?

Proposition 27 showed the characterization of the generalized reaping numbers via graph coloring properties. Can we find a similar correspondence for other cardinal invariants? That is:

Question 35. For some cardinal invariant $\mathfrak{x}$ and its generalization $\mathfrak{x}_\kappa$ for every cardinal κ, does there exist a graph X_λ for each cardinal λ and a coloring property P such that $P(\kappa, X_\lambda)$ is equivalent to $\mathfrak{x}_\kappa < \lambda$?

Stationary sets are positive sets with respect to the nonstationary ideal. Therefore, in the same way as our stationary list coloring, we can consider I^+-list coloring for an arbitrary ideal I.

Question 36. What can be said about the corresponding I^+-list coloring property for an ideal I?

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